STRENGTH OF MATERIALS

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A TEXTBOOK
COVERING THE SYLLABUSES OF
THE B.Sc. (Eng.), I.C.E. and I.Mech.E.
EXAMINATIONS IN THIS SUBJECT

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SEVENTH EDITION

LONDON
SIR ISAAC PITMAN & SONS, LTD.

PREFACE

TO THE SEVENTH EDITION

In this edition a few minor alterations and additions have been made to the worked examples, and in view of its rapidly increasing importance a chapter on photo-elasticity has been added. The material in this chapter will serve to give the reader the basis of this method of stress analysis.

The author wishes to thank the editor of *The Engineer* for permission to use Figs. 156, 157, and 159, and, as before, he takes this opportunity of expressing his indebtedness to Messrs. Thos. Firth and John Brown, Ltd., Sheffield, for permission to reproduce the table on comparative hardness values and to Messrs. High Duty Alloys, Ltd., for the vast amount of matter relating to the mechanical properties of their aluminium alloys.

F. V. WARNOCK

PREFACE

This book has been written for engineering students, and although its main purpose is to assist those students working for the degrees of the University of London and the final examinations of the various professional engineering institutions, it will be found to cover the course of other universities. It is also hoped that it will prove useful to many practical engineers.

In arriving at the various formulae the calculus has been used, as nearly all engineering students have a working knowledge of this branch of mathematics.

The book contains a liberal supply of worked examples taken from the recent examination papers of the University of London and the Institutions of Civil and Mechanical Engineers. At the end of each chapter a large number of exercises will be found, taken from the same sources.

The chapter dealing with the deflection of beams has been influenced to some extent by Prof. J. H. Smith, D.Sc., and the reader is advised to become familiar with the method of treating deflection and slope from a knowledge of the bending-moment diagram, as in the majority of cases this method greatly simplifies the calculations.

In many cases reference is made to important researches bearing on the subject, and in each case it will be found that the reference simplifies the access to the original papers. In the last chapter the fatigue of metals has been dealt with rather briefly, but the reader is referred to a comprehensive work on the subject. It will be found, however, that the synopsis contains the most important of the generally accepted results.

The author wishes to thank the various professional institutions for permission to make use of a great amount of material taken from their journals, and the following firms who have kindly supplied blocks or photographs: W. and T. Avery, Ltd., Birmingham; A. Macklow-Smith, London; Joshua Buckton & Co., Ltd., Leeds; Cambridge Instrument Co., Ltd., London; Messrs. W. G. Pye & Co., Cambridge; Messrs. Kayser Ellison & Co., Ltd., Sheffield; and the editor of *Engineering* for the description of the Lamb extensometer.

The author is also greatly indebted to Mr. J. McKeown, M.Sc., and to Mr. S. Clarke for their valuable assistance in the production of the book.

It is hoped that the errors in this edition will be comparatively few—on account of the huge amount of arithmetical work it is too much to hope that there are none. The author will be grateful to receive intimation of any errors and helpful suggestions will be appreciated.

F. V. WARNOCK

BELFAST
1927

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STRENGTH OF MATERIALS

CHAPTER I

SIMPLE STRESSES AND STRAINS

1. **Introduction.** When an engineer undertakes the design of a machine part or portion of a structure, it is essential that he should fully realize the various forces for which allowances have to be made. He then requires to have at his disposal formulae which will enable him to proportion the design so that fracture will not take place when the part is subjected to the estimated forces. The formulae which follow have been largely arrived at by making use of well-known facts in the study of statics, dynamics, and mathematics. In the derivation of these formulae assumptions have been made, which are very often not exactly realized in practice. The results obtained by using such formulae should be compared, therefore, wherever possible, with those obtained experimentally.

The reader is advised, when attacking a particular problem in design, first of all to make himself perfectly familiar with the conditions under which the part has to work. Secondly, before making use of a particular formula, to be familiar with the assumptions made in obtaining the formula, and thus see how closely theoretical and working conditions agree. Finally, in making use of the formula, he should incorporate common sense and the results of experience.

2. **Load.** The external forces acting on a piece of material constitute what is called the "load." The following are a few of the forces met with in practice—

(*a*) A force, due to a load not in motion; an example of which is a load hanging from a crane chain, without being raised or lowered.

(*b*) An inertia force, due to change in velocity of a mass. This is met with in engine practice and is the result of change in velocity of the reciprocating parts.

(*c*) A centrifugal force, which is met with in pulleys and flywheels, and is due to the tendency of a rotating mass to get away from its centre of rotation.

(*d*) A frictional force, resulting from the application of a brake on a drum.

(*e*) A force due to expansion or contraction, which is met with in boiler furnaces.

Forces such as those above mentioned often act at the end of an arm, and cause bending or twisting in the material.

3. **Stress.** No material is perfectly rigid, therefore the application of a load to a piece of material causes a deformation. This deformation may be large and easily measured as in the case of a piece of rubber 1 sq. in. cross-section carrying a pull of 100 lb., or it may be small and require a delicate measuring device where a steel bar of the same cross-section carries a pull of 10 tons. In all cases internal forces are called into play in the material to resist the load and are referred to as "stresses." The intensity of the stress is estimated as the force acting on unit area of cross-section, and is expressed in such units as tons per sq. in., lb. per sq. in., etc.

4. **Tensile Stress.** An example of a body stressed in this manner is shown in Fig. 1, which represents a uniform vertical bar held at its upper end and carrying an axial load W. If the bar is cut at CB, it is easily seen that a force P, acting in the opposite direction to W, and equal in magnitude to W, is required to hold the lower portion in equilibrium. In the normal state of the bar, the force P is supplied by the internal forces in the material. If the section CB is taken at any point between the ends of the bar a similar condition exists (neglecting the weight of the portion of the bar below the section).

The intensity of tensile stress at CB is given by $f_t = \frac{W}{A}$ where A is the cross-sectional area of the bar.

5. **Compressive Stress.** Fig. 2 represents a uniform vertical bar carrying a load W, which in this case causes a stress of opposite nature to that discussed in par. 4. The line of action of the load is again assumed to be axial. Considering a section CB, it is seen that a force P equal in magnitude to W, and acting in the opposite direction to W, is required for equilibrium. Again, neglecting the weight of the portion of the bar above CB, the intensity of the compression stress at CB is given by—

$$f_c = \frac{W}{A}$$

where A is the cross-sectional area of the bar.

6. **Shear Stress.** A stress of this nature is said to exist on a section of a body if on opposite faces of the section equal and opposite parallel forces exist. Let a rectangular block of metal of cross-sectional area A be soldered to a heavy mass of iron and suppose a force W to be applied, acting as shown in Fig. 3, now consider the section CB. The upper portion H exerts a force W on the face of the lower portion K, which portion in turn exerts an equal and opposite force on the face

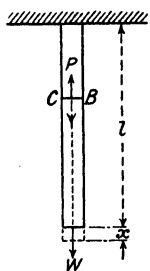



FIG. 1

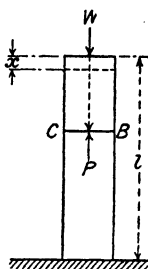



FIG. 2

of the upper portion H. The intensity of the shear stress on the section at CB is given by

$$f_s = \frac{W}{A}$$

In each of the three cases considered the force is assumed to be distributed uniformly over the surface. If this is not the case then the intensity of stress at a point in the surface is taken to be equal to the limiting ratio of $\frac{\delta W}{\delta A}$ when each is reduced indefinitely where δW is the force acting on the very small area δA.

7. **Strain.** Strain is a measure of the deformation produced by the application of the external forces.

(*a*) In Fig. 1 it will be observed that the deformation is an elongation of the amount x, and if l is the initial length of the bar, then the tensile strain is given by

$$e_t = \frac{x}{l}$$

or is the elongation per unit of length.

(*b*) In Fig. 2 the deformation is shown to be a shortening of

the bar by the amount x, and if l is the unloaded length, the compressive strain is given by

$$e_c = \frac{x}{l}$$

or the shortening per unit of length.

(*c*) The state of deformation produced by shear is shown by the dotted lines in Fig. 3. The movement x of the corner is exceedingly small so that EFL may be regarded as a right-angled triangle. The shear strain is given by

$$e_s = \frac{x}{l} = \tan \theta.$$

(*d*) Another kind of strain is encountered if a solid such as a cube be subjected to forces on each face. Fig. 4 shows a cube

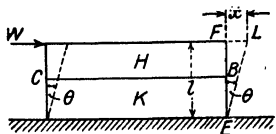



Fig. 3

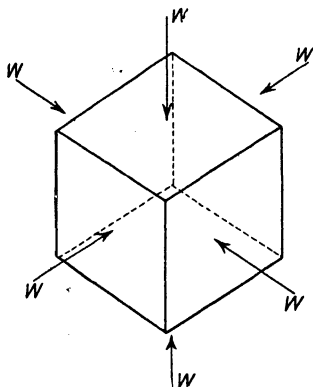



Fig. 4

subjected to equal compressive normal forces on each face. (This state of loading would be obtained if the cube was lowered to a great depth in a liquid.) The result of the forces on each face will be a shortening of each edge of the cube and hence a reduction in volume. The cubical or volume strain is given by

$$e_v = \frac{\text{change in volume}}{\text{original volume}}$$

or change in volume per unit of volume.

If l is the length of the edge of the unstrained cube and x the decrease in length of each edge, then the change in volume is given by $3\ l^2x - 3\ lx^2 + x^3$. Now since x is very small, terms containing x^2 or higher powers may be neglected, hence the change in volume is approximately $3\ l^2x$. The volume strain is therefore approximately $\frac{3x}{l}$, which is three times the linear strain of the edge.

It is important to note that in each case the strain is a ratio and thus non-dimensional.

Let the dimensions of a bar be x, y and z, and suppose the bar to be loaded in such a manner that the corresponding strains in the directions of the above dimensions are e_x, e_y, and e_z. Any of these strains may be positive or negative, depending on whether they are elongations or contractions.

The new dimensions of the bar are $x(1 + e_x)$, $y(1 + e_y)$ and $z(1 + e_z)$.

The volume of the unstrained bar $= xyz$

" " " strained " $= x(1 + e_x)\,y(1 + e_y)$
$z(1 + e_z) = xyz(1 + e_x + e_y + e_z)$

if the product of two or more strains be neglected.

The change in volume of the bar

$$= xyz(e_x + e_y + e_z)$$

and the volume strain $= \dfrac{\text{change in volume}}{\text{original volume}} = e_v$ is given by

$$e_v = e_x + e_y + e_z.$$

8. **Elasticity.** A material is said to be perfectly elastic if the strain due to loading disappears with the removal of the load, and also if the strain for a given value of load during the unloading process is equal to the strain for the same value of load during the loading process.

A limiting value of load will be found at which the strain does not completely disappear with the removal of the load. The value of the stress corresponding to this load is called the "Elastic Limit," and the residual strain is referred to as a permanent set.

9. **Hooke's Law.** It was discovered by Hooke that if a material be loaded without exceeding the elastic limit, then the deformation produced is proportional to the load producing it. Now the stress caused by the load is proportional to the load, and the corresponding strain is proportional to the deformation, therefore the stress is proportional to the strain, or

$$\frac{\text{stress}}{\text{strain}} = \frac{W}{A} \times \frac{l}{x} = k\frac{W}{x} \text{ where } k = \frac{l}{A}$$

and since x is proportional to W, then $k\dfrac{W}{x}$ is equal to a constant, i.e. $\dfrac{\text{stress}}{\text{strain}} = N$.

This constant is called the modulus of elasticity, and its magnitude will depend on the material and on the nature of the stress and strain dealt with. Since stress is a force per unit of area, and strain is a number, the modulus of elasticity will also be a force per unit area.

10. **The Elastic Constants.** When a body is subjected to simple tension or compression the modulus of elasticity is usually called Young's modulus, and it is invariably denoted by the letter E. In the case of a body subjected to pure shear only the modulus is called the shear or rigidity modulus, and is usually denoted by the letter C.

The bulk or volume modulus corresponds to the strain arising from a state of loading such as is indicated by Fig. 4, and is denoted by K.

If f_t and f_c denote the tensile or compressive stress in a body subjected to pure tension or compression, and e_t and e_c the corresponding strains, then

$$E = \frac{f_t}{e_t} = \frac{f_c}{e_c}$$

Also if f_s is the shear stress in a body subjected to pure shear and e_s the corresponding strain, then

$$C = \frac{f_s}{e_s}$$

In Fig. 4 if f_v denotes the intensity of the normal stress on each face of the cube and e_v the corresponding strain, then

$$K = \frac{f_v}{e_v}$$

These constants are related to one another, which relationship will be obtained in a later chapter.

From a knowledge of the relationship, just obtained, between stress, strain and modulus of elasticity, it will be found that the attack on a problem on strength of materials develops along the following lines—

(1) Assume a deformation (this should be of as simple a character as possible).

(2) Calculate the strains.

(3) Multiply the strains by the elastic moduli and so obtain the stresses.

(4) Multiply the stresses by the areas over which they act,

equate to the external forces, and so proceed with the solution of the problem.

Numerous examples of the use of this method of attack will be shown by means of worked examples, and the reader is advised to study each one carefully.

EXAMPLE 1.

Define " stress," " strain," and " modulus of elasticity " in general terms. A metal rod of circular section, 1 in. in diameter, is subjected to stress in a tension-testing machine. It is found that the total extension over a length of 8 in. is 151 scale divisions of an extensometer for a pull of 9,000 lb., the unit of the scale being $\frac{1}{50,000}$ of an inch. Calculate the stress, the strain and the modulus of elasticity for this rod. (London University, 1913.)

Area of cross-section of rod $= 0{\cdot}7854 \times 1^2 = 0{\cdot}7854$ sq. in.

$$f_t = \frac{W}{A} = \frac{9000}{0{\cdot}7854}$$

$$= 11{,}460 \text{ lb./sq. in.}$$

Extension on 8 in. length = scale divisions $\times$ unit of scale

$$= 151 \times \frac{1}{50{,}000} \text{ in.}$$

$$e_t = \frac{x}{l} = \frac{151}{50{,}000 \times 8}$$

$$= 0{\cdot}0003775.$$

$$E = \frac{f_t}{e_t} = \frac{11{,}460}{0{\cdot}0003775}$$

$$= 30{\cdot}37 \times 10^6 \text{ lb./sq. in.}$$

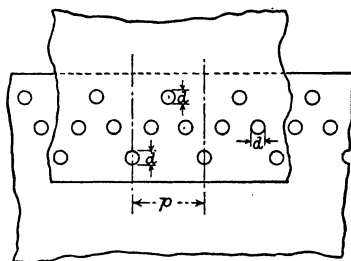



FIG. 5

EXAMPLE 2.

Design a treble-riveted lap joint in which the pitch of the rivets in the outer rows is twice the pitch of the rivets in the inner row, for plates ¾ in. in thickness. Determine the pitch and diameter of the rivets so that the tensile stress in the plates is 6 tons per sq. in. and the shearing stress in the rivets is 4½ tons per sq. in. (Lond. Univ., 1912.)

The arrangement of the joint is shown in Fig. 5.

The diameter of the rivets is found from the formula

$$d = 1{\cdot}2\sqrt{t} \text{ where } t = \text{thickness of plates in inches.}$$

$$\therefore d = 1{\cdot}2\sqrt{0{\cdot}75}$$

$$\doteq 1 \text{ in.}$$

Let n = number of rivets in strip of width equal to p,
where p = pitch of rivets.

Strength of plate against tearing

$$= (p - d)t \times f_t$$
$$= (p - 1) \times \tfrac{3}{4} \times 6$$
$$= 4\tfrac{1}{2}p - 4\tfrac{1}{2}.$$

Strength of rivets against shearing

$$= \frac{\pi}{4} d^2 \times f_s \times n$$
$$= 0{\cdot}78 \times 1 \times 4{\cdot}5 \times 4$$
$$= 14{\cdot}15 \text{ tons.}$$

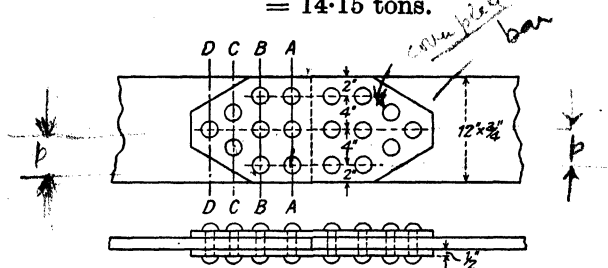



FIG. 5A

Since joint is equally strong in tension and in shear,

$$4\tfrac{1}{2}p - 4\tfrac{1}{2} = 14{\cdot}15$$
$$\therefore p = 4{\cdot}14 \text{ in., say 4 in. pitch.}$$

EXAMPLE 3.

Two lengths of bar 12 in. wide and $\tfrac{3}{4}$ in. thick are to be connected by a double cover butt joint. The diameter of the rivet holes is 1 in. Design a suitable joint. Tensile stress, 7 tons/sq. in. Shear stress, 5 tons/sq. in. Bearing stress, 9 tons/sq. in. (Lond. Univ., 1916.)

The rivets in a joint of this type are usually arranged as shown in Fig. 5A, the bar is then only weakened by one rivet hole.

Strength of joint against tearing through the outside rivet $= (12 - 1) \tfrac{3}{4} \times 7 = 57{\cdot}75$ tons.

Shear strength of one rivet $= \left\{\frac{\pi}{4} \times (1)^2 \times 5\right\} 2$ since rivets are in double shear

$$= 7{\cdot}85 \text{ tons.}$$

$\therefore$ Number of rivets required for shear $= \dfrac{57 \cdot 75}{7 \cdot 85} = 7 \cdot 4$, say 8.

Bearing strength of each rivet $= 9 \times 1 \times \frac{3}{4} = 6 \cdot 75$ tons.

$\therefore$ Number of rivets required for bearing $= \dfrac{57 \cdot 75}{6 \cdot 75} = 8 \cdot 55$, say 9.

The joint would therefore require 9 rivets in each bar, and would be arranged as shown in Fig. 5A.

Each cover plate might be taken to be half the thickness of the bars, but in practice it is usual to make the cover plates $\frac{5}{8}$ the thickness of the bars.

$\therefore$ Thickness of each cover plate $= \frac{5}{8} \times \frac{3}{4} \doteqdot \frac{1}{2}$ in.

It is of interest to further extend this example and find the force necessary to rupture the joint at *DD*, *CC*, etc., and hence obtain the efficiency of the joint.

The force necessary to rupture the joint at *DD* has already been found to be 57·75 tons.

Suppose bar to tear at *CC* and rivet at *DD* to shear. Total force required $= (12 - 2) \frac{3}{4} \times 7 + 7 \cdot 85 = 60 \cdot 35$ tons.

Suppose bar to tear at *BB* and rivets at *CC* and *DD* to shear. Total force required $= (12 - 3) \frac{3}{4} \times 7 + 7 \cdot 85 \times 3 = 70 \cdot 8$ tons.

Suppose bar to tear at *AA* and rivets at *BB*, *CC* and *DD* to shear. Total force required $= (12 - 3) \frac{3}{4} \times 7 + 6 \times 7 \cdot 85 = 94 \cdot 35$ tons.

Suppose cover plates ruptured along *AA*. Total force required $= 7 (12 - 3) \times 2 \times \frac{1}{2} = 63$ tons.

From this it is clear that the weakest section is at *DD*.

$$\text{The efficiency of the joint} = \frac{\text{least strength of the joint}}{\text{strength of solid plate}} = \frac{57 \cdot 75}{63} = 91 \cdot 6 \text{ per cent.}$$

EXAMPLE 4.

The diameter of the piston of a Diesel engine is 310 mm., and the maximum compression pressure in the cylinder is 500 lb./sq. in. The cylinder is held by four bolts whose effective diameter is 2 in. and length 35 in. Estimate the maximum tensile stress in each bolt, and the elongation of each bolt. $E = 30 \times 10^6$ lb./sq. in.

Maximum force exerted on each bolt

$$= \left(\frac{310}{25 \cdot 4}\right)^2 \times 0 \cdot 7854 \times \frac{500}{4} = 14{,}630 \text{ lb.}$$

Maximum stress in each bolt $= f_t$

$$= \frac{14,630}{0\cdot7854 \times 4}$$

$$= 4656 \text{ lb./sq. in.}$$

Elongation of each bolt $= \frac{f_t \times l}{E} = \frac{4656 \times 35}{30 \times 10^6}$

$$= 0\cdot005433 \text{ in.}$$

11. **Bar of Uniform Strength.** Fig. 6 represents a tie-bar rigidly secured at its upper end, it is required to find the shape of the bar in order that the tensile stress at all cross-sections may be constant.

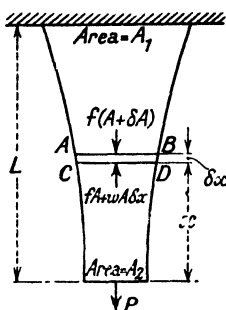



FIG. 6

Consider a small strip of the bar of thickness δx between the sections AB and CD distant x from the free end.

Let A be the area of the section at CD and $A + \delta A$ the area of the section at AB.

Since the strip is in equilibrium, then

The total force acting upwards
= Total forces acting downwards,

i.e. (tensile stress $\times$ area of section at AB = tensile stress $\times$ area of section at CD + weight of slice of thickness δx).

If w = weight of unit volume of the bar and
f = tensile stress at any cross-section,

then

$$f(A + \delta A) = fA + wA\delta x, \text{ or}$$

$$f\left(A + \frac{\delta A \,.\, \delta x}{\delta x}\right) = fA + wA\delta x$$

$$f\frac{\delta A \,.\, \delta x}{\delta x} = wA\delta x$$

$$f\frac{\delta A}{\delta x} = wA$$

$$\therefore \frac{\delta A}{A} = \frac{w}{f}\,\delta x$$

In the limit, when the slice is infinitely thin, this reduces to

$$\frac{dA}{A} = \frac{w}{f} \cdot dx$$

Integrating each side, we get

$$\int_{A_2}^{A} \frac{dA}{A} = \frac{w}{f} \int_{o}^{x} dx$$

$$\log_e \frac{A}{A_2} = \frac{w}{f} x$$

$$\text{or } A = A_2 e^{\frac{w}{f} x}$$

where e is the base of the Naperian logarithm = 2·718,

$$\text{also } A_1 = A_2 e^{\frac{w}{f} L}$$

12. **Extension of a Taper Rod.** A rod of length L tapers uniformly as shown by Fig. 7, and carries an axial tensile load W.

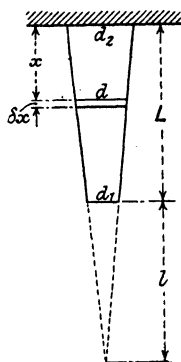
FIG. 7

Let A_1 = the smallest cross-sectional area,

d_1 = the diameter of the area A_1 and

f_1 = the tensile stress on A_1,

then A_2, d_2 and f_2 are the corresponding values for the largest diameter.

Consider a small length of the rod δx, distant x from the larger end, at which the cross-sectional area is A, the diameter d, and the tensile stress f.

If δu is the increase in length of the small length δx,

$$\text{then } \delta u = \frac{f}{E} \delta x$$

$$\text{but } fA = f_1 A_1 = W$$

$$\therefore \delta u = \frac{f_1}{E} \frac{A_1}{A} \delta x \qquad \qquad (1)$$

$$\text{Now } \frac{A_1}{A} = \frac{d_1^2}{d^2} = \frac{l^2}{(l + L - x)^2}$$

Hence by substitution in (1) $\delta u = \frac{f_1}{E} \cdot \frac{l^2}{(l + L - x)^2} \delta x$

or $du = \frac{f_1}{E} \cdot \frac{l^2}{(l + L - x)^2} dx$ when δx is infinitely small . (2)

Fig. 8

The total extension $u = \int du$.

$$= \int_0^L \frac{f_1 l^2}{E} \frac{dx}{(l + L - x)^2}$$

$$= \frac{f_1 l^2}{E} \left| \frac{1}{l + L - x} \right|_0^L = \frac{f_1 l}{E} \cdot \frac{L}{l + L}$$

$$= \frac{WL}{A_1 E} \cdot \frac{l}{l + L}$$

$$\text{or } u = \frac{WL}{A_1 E} \cdot \frac{d_1}{d_2} \quad . \quad . \quad . \quad . \quad (3)$$

For a parallel bar $\frac{d_1}{d_2} = 1$, and the expression reduces to $\frac{WL}{A_1 E}$ which is the expression for the extension of a parallel bar carrying a tensile load W.

13. **Compound Bars and Columns.** A short compound column, composed of a steel tube fitting loosely inside a copper tube, and carrying an axial load W, is shown in Fig. 8.

It is evident that the strain in each tube will have the same value, but since E is different for each material, the stress will not have the same value for each material.

Let A_c = cross-sectional area of copper tube

A_s = ,, ,, ,, steel ,,

W_c = load carried by the copper tube

W_s = ,, ,, ,, steel ,,

E_c and E_s being the value of Young's modulus for copper and steel, and f_c and f_s being the value of the corresponding stresses in the tubes.

If l is the unloaded length of the column, and x is the amount each tube shortens.

$$\text{Strain in each} = \frac{x}{l}$$

$$but\ \frac{x}{l} = \frac{f}{E}$$

$$\therefore \frac{f_s}{E_s} = \frac{f_c}{E_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also $$W = W_s + W_c = f_sA_s + f_cA_c \quad . \quad . \quad . \quad (2)$$

and from (1)

$$W = f_sA_s + f_s\frac{E_c}{E_s}A_c$$

$$\therefore f_s = \frac{WE_s}{A_sE_s + A_cE_c}$$

$$\text{similarly } f_c = \frac{WE_c}{A_sE_s + A_cE_c}$$

14. **Temperature Stresses.** When the temperature of a metal changes there is a corresponding change in dimensions. If this change in dimensions is prevented then a stress is set up in the metal.

The linear expansion of the metal is proportional to the change in temperature, and this expansion per unit of length per unit change in temperature is called the linear co-efficient of expansion of the metal.

Suppose a bar be held in such a manner that linear change of dimensions is prevented when heat is applied.

Let l = length of bar

T = change in temperature

a = linear co-efficient of expansion

If the bar was free to expand the increase in length

$$= aTl$$

Hence the strain set up due to expansion being prevented

$$= \frac{aTl}{l}$$

$$= aT$$

If f = stress due to the above strain,

E = Young's modulus for the material of the bar.

$$\text{Since } \frac{\text{stress}}{\text{strain}} = \text{modulus of elasticity}$$

$$\therefore \frac{f}{aT} = E$$

or $$f = aTE$$

EXAMPLE 5.

A steam pipe is 100 ft. long at a temperature of 15° C. Steam at 180° C. is passed through the pipe. What is its length when free to expand ? Suppose expansion to be prevented, what stress is induced in the material ? $E = 6{,}000$ tons/sq. in. $a = 0{\cdot}000012$ per ° C.

Change in temperature $= 180 - 15$

$= 165°$ C.

Increase in length $= aTl$

$= 0{\cdot}000012 \times 165 \times 100$

$= 0{\cdot}198$ ft.

Strain due to prevention of expansion

$$= \frac{aTl}{l}$$

$= aT$

$= 0{\cdot}000012 \times 165$

Stress induced in material $= aTE$

$= 0{\cdot}000012 \times 165 \times 6000$

$= 11{\cdot}88$ tons/sq. in.

EXAMPLE 6.

Three rods, each initially of ½ sq. in. cross-section and 5 ft. long, support a load of 10 tons. The centre rod is made of steel and the outer ones of copper. If the temperature of the rods is increased by 100° C. and the rods are so adjusted that they are extended equal amounts, estimate the load carried by each rod. $E_s = 30 \times 10^6$ lb./sq. in. $E_c = 12 \times 10^6$ lb./sq. in. $a_s = 0{\cdot}000012$ per ° C. $a_c = 0{\cdot}0000185$ per ° C.

Let x = the common extension.

Then aTl = extension due to heating,

and $\left(\frac{x}{l} - aT\right)$ = strain due to load carried.

$$\text{Stress in each rod} = \left(\frac{x}{l} - aT\right) E \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Load carried by rod} = \left(\frac{x}{l} - aT\right) EA \quad . \quad . \quad . \quad (2)$$

where A = cross-sectional area of rod.

$$\text{Total load} = W = \sum \left(\frac{x}{l} - aT\right) EA. \quad . \quad . \quad . \quad (3)$$

$$\therefore 22{,}400 = \left(\frac{x}{l} - 0{\cdot}000012 \times 100\right) 30 \times 10^6 \times \tfrac{1}{2}$$
$$+ \left(\frac{x}{l} - 0{\cdot}0000185 \times 100\right) 12 \times 10^6 \times \tfrac{1}{2} \times 2$$

$$\frac{x}{l} = \frac{22{,}400 + \frac{100}{2}(0{\cdot}000012 \times 30 + 0{\cdot}0000185 \times 24)10^6}{\frac{1}{2} \times 10^6 (30 + 24)}$$
$$= 0{\cdot}002318.$$

Load carried by steel rod

$= (0{\cdot}002318 - 0{\cdot}000012 \times 100)\ 30 \times 10^6 \times \frac{1}{2}$

$= 0{\cdot}001118 \times 15 \times 10^6$

$= \underline{16{,}770 \text{ lb.}}$

Each copper rod carries a load

$= (22{,}400 - 16{,}770)\ \frac{1}{2}$

$= 5630 \times \frac{1}{2}$

$= \underline{2815 \text{ lb.}}$

EXAMPLES I

1. A flat mild steel bar of rectangular cross-section, 3 in. × $\frac{3}{4}$ in., carries an axial pull of 12 tons. Estimate the tensile stress in the bar and the elongation if the unloaded length is 20 ft. $E = 30 \times 10^6$ lb./sq. in.
Ans., 5·33 tons/sq. in., 0·0956 in.

2. A cast-iron column, 15 ft. long, has a hollow circular cross-section. The external diameter is 12 in., and the thickness of the material $1\frac{1}{4}$ in. The column is subjected to a compressive stress, of 4 tons per sq. in., by a central load on top of the column. If Young's modulus for cast iron is 8,000 tons per sq. in., estimate the value of the load and the amount the column shortens under the load. *Ans.*, 168·86 tons, 0·09 in.

3. A tie-bar $1\frac{1}{2}$ in. in diameter carries a load which causes a tensile stress of 8,000 lb. per sq. in. The bar is fastened to a cast-iron bracket which is held by four bolts. Find the diameter of the bolts at the bottom of the threads if the stress is limited to 5,000 lb. per sq. in. *Ans.*, 0·948 in.

4. If the ultimate shear stress for mild steel is 55,000 lb./sq. in., find the force required to punch a $\frac{3}{4}$-in. hole in a mild steel plate $\frac{1}{2}$ in. thick. What is the compressive stress on the punch? *Ans.*, 28·94 tons, 65·48 tons/sq. in.

5. A vertical tie-bar has a cross-section whose diameter is 2 in., at which the tensile stress is 6,000 lb./sq. in. If the stress in the bar is to be constant at all cross-sections, find the diameter of the section at a point 20 ft. above the section whose diameter is 2 in. Density of material 0·29 lb./cu. in.

Ans., 2·012 in.

6. A short steel tube 4 in. internal diameter and ½ in. thick is surrounded loosely by a brass tube of the same length and thickness. The tubes carry an axial thrust of ½ a ton. Estimate the load carried by each tube, and the amount each tube shortens. Length 3½ in.

Ans., $W_S = 756{\cdot}2$ lb., $W_B = 363{\cdot}8$ lb., $1{\cdot}25 \times 10^{-5}$ in.

7. Two vertical rods are each rigidly fastened at the upper end at a distance of 24 in. apart. Each rod is 10 ft. long and ½ in. in diameter. A horizontal cross-bar connects the lower ends of the rods and on it is placed a load of 1,000 lb. so that the cross-bar remains horizontal. Find the position of the load on the cross-bar and estimate the stress in each rod. One rod is of wrought iron for which $E = 28 \times 10^6$ lb./sq. in., and the other of bronze for which $E = 9 \times 10^6$ lb./sq. in.

Ans., $f_{WI} = 3{,}855$ lb./sq. in. $f_B = 1{,}239$ lb./sq. in., 5·8 in. from the w.i. bar.

8. A steel rail is 30 ft. long at a temperature of 15° C. Estimate the elongation when the temperature increases to 85° C. If no allowance is made for expansion calculate the stress in the rail. $a = 0{\cdot}000012$ per ° C. $E = 30 \times 10^6$ lb./sq. in. *Ans.*, 0·302 in. 25,200 lb./sq. in.

9. A weight of 20 tons is supported by three short pillars each 1 sq. in. in section. The centre pillar is of steel and the two outer ones of copper. The pillars are so adjusted that at a temperature of 15° C. each carries one-third of the total load. The temperature is then raised to 115° C. Estimate the stress in each pillar at 15° C. and at 115° C. $E_s = 30 \times 10^6$ lb./sq. in. $E_c = 12 \times 10^6$ lb./sq. in. $a_s = 12 \times 10^{-6}$ per ° C. $a_c = 18{\cdot}5 \times 10^{-6}$ per ° C.

Ans., 6·66 tons/sq. in. $f_c = 8{\cdot}6$ tons/sq. in. $f_s = 2{\cdot}8$ tons/sq. in.

10. A mild steel bar ¾ in. diameter and 12 in. long is placed inside a tube having an external diameter of 1 in. and an internal diameter of ¾ in. The combination is then subjected to an axial thrust of 5 tons. The modulus of elasticity of the metal of the tube is 11,500,000 lb./sq. in., and of the steel 30,000,000 lb./sq. in. Find : (*a*) the stress in the tube and in the rod, (*b*) the shortening of the rod, (*c*) the work done in compression. (Lond. Univ., 1915.)

Ans., $f_s = 8{\cdot}72$ tons/sq. in. $f_{tube} = 3{\cdot}34$ tons/sq. in. ; 0·007812 in. ; 43·75 in. lb.

11. A steel rod, circular in section, tapers from 1 in. diameter to ½ in. diameter in a length of 24 in. Find how much this length will increase under a pull of 2 tons if Young's Modulus is 30×10^6 lb./sq. in. (A.M.I.Mech.E., 1926.)

Ans., 0·009126 in.

12. A bar of copper, 1·5 in. diameter, is completely enclosed in a steel tube, 2·5 in. external diameter. A pin, 0·75 in. diameter, is fitted transversely to the axis of the bar near each end, to secure the bar to the tube. Calculate the intensity of shear stress induced in the pins when the temperature of the whole is raised 100° F. $E_c = 6{\cdot}5 \times 10^3$ tons/sq. in., $E_s = 13 \times 10^3$ tons/sq. in. $a_c = 9{\cdot}5 \times 10^{-6}$ per ° F. $a_s = 6{\cdot}2 \times 10^{-6}$ per ° F. (A.M.I.Mech.E., 1924.)

Ans., 3·48 tons/sq. in.

13. A bar of steel is 24 in. long, and the two ends are respectively 1¼ in. and 1½ in. in diameter, and each is 6 in. in length, the middle portion being

1 in. diameter. Determine the length of the bar when subjected to an axial compression load of 11 tons. $E = 12{,}500$ tons/sq. in. (A.M.I.Mech.E., 1924.)

Ans., 23·979 in.

14. Design a riveted joint for jointing two lengths of flat tie-bar. Double cover plates. Find the efficiency of the joint. Load 80 tons. Diameter of rivets $1\frac{1}{8}$ in. Clearance in rivet holes 0·05 in. Tensile stress not to exceed 5 tons/sq. in. Shearing stress not to exceed 4 tons/sq. in. Bearing stress not to exceed 6 tons/sq. in. The width should be about 15 times the thickness. (Lond. Univ., 1915.) *Ans.*, 88·8 per cent.

15. Two thick plates made of an aluminium alloy are held together in contact by copper bolts. Find the increase in the tensile stress (in tons per square inch) in the bolts due to a rise of temperature from 60° F. to 80° F., neglecting any compressive strain in the aluminium alloy. Modulus of direct elasticity for copper = 18×10^6 lb. per sq. inch. Co-efficient of expansion of copper per ° F. = 0·0000093. Co-efficient of expansion of aluminium alloy per ° F. = 0·0000126. (A.M.I.C.E., 1925.)

Ans., 0·534 tons/sq. in.

16. If a tension test bar is found to taper uniformly from $(D - a)$ inches diameter to $(D + a)$ inches, prove that the error involved in using the mean diameter to calculate Young's modulus is $\left(\frac{10a}{D}\right)^2$ per cent. (Lond. Univ., 1936.)

17. An elastic packing piece is bolted between a rigid rectangular plate and a rigid foundation by two bolts pitched 10 in. apart and symmetrically placed on the long centre line of the plate which is 15 in. long. The tension in each bolt is initially 4,000 lb., the extension of each bolt is 0·0005 in. and the compression of the packing piece is 0·02 in. If one bolt is further tightened to a tension of 5,000 lb. determine the tension in the other bolt. (Lond. Univ., 1943.)

Ans., 4140 lb.

18. A square rigid base-plate 20 in. × 20 in. bears a column which applies a central load of 5 tons. The base-plate is held down to a rigid foundation by four bolts placed symmetrically at corners of a square of 16 in. side. Between the base-plate and the foundation there is a sheet of elastic packing. While the load is carried, the bolts are tightened up to a tension of 0·5 ton, the extension of the bolts being half the compression of the packing due to the load and the tension of the bolts. If the line of action of the 5-ton load shifts 2 in. in a direction parallel to a side of the base-plate, find the new tensions in the bolts. Neglect the bolt holes in the packing. (Lond. Univ., 1944.)

Ans., 0·5867 tons, 0·4133 tons.

CHAPTER II

COMPOUND STRESSES AND STRAINS

15. **Normal and Shear Stresses** on an oblique section of a bar subjected to direct loading. In Fig. 9 a bar of cross-sectional area A is shown carrying an axial tensile load W.

Let CD be a section inclined at an angle θ to the cross-section BD. Considering the portion of the bar above CD, it is seen that the force W is inclined to the direction of CD, and therefore can be resolved into components, a component N perpendicular to CD, and a component S parallel to CD.

The normal component N introduces a tensile stress on the section at CD, and the component S introduces a shear stress on the section. Let f_n and f_s be the stresses due to N and S respectively, and let $f = \dfrac{W}{A}$.

$$\text{Area of section at } CD = \frac{\text{area of section at } BD}{\cos\theta} = \frac{A}{\cos\theta}$$

$$\text{also } N = W\cos\theta$$

$$\text{and } S = W\sin\theta$$

$$\therefore f_n = \frac{N}{\text{area at } CD} = \frac{W\cos\theta}{\dfrac{A}{\cos\theta}} = \frac{W}{A}\cos^2\theta \qquad . \qquad . \quad (1)$$

$$f_s = \frac{S}{\text{area at } CD} = \frac{W\sin\theta}{\dfrac{A}{\cos\theta}} = \frac{W}{A}\sin\theta\cos\theta \qquad . \quad (2)$$

From (1) f_n will be maximum when $\cos\theta$ is maximum, i.e $\theta = 0$. The maximum value of f_n is then $= \dfrac{W}{A} = f$.

From (2) $f_s = \dfrac{W}{2A}\sin 2\theta$

$$= \frac{f}{2}\sin 2\theta$$

f_s will therefore be a maximum when sin 2θ is a maximum, i.e. when $2\theta = 90°$, or $\theta = 45°$. The maximum value of f_s is then equal to $\frac{f}{2}$.

From this result it is seen that a direct stress introduces a shear stress of half its intensity on planes inclined at 45° to the plane carrying the direct stress. It follows, therefore, that if a material is such that *its shear strength is less than half its tensile strength, then the material will fail by shear when subjected to a tensile load.* Similarly, if the direction of W be reversed, it can be shown that if the shear strength of a material is less than half its compressive strength, then the material will fail by shear when subjected to a compressive load.

This explains the peculiar fracture of mild steel when tested, to destruction, in tension, and that of cast iron when tested, to destruction, in compression. A further reference to these fractures will be found in the chapter dealing with the testing of materials.

FIG. 9

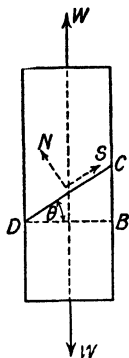
EXAMPLE 1.

A short cast-iron block of rectangular cross-section is subjected to an axial thrust of 5 tons. Estimate the compressive and shear stresses on a section inclined at 60° to the direction of thrust. Neglecting the effects of friction, estimate the maximum shear stress in the block.

Area of cross-section perpendicular to thrust = 1 sq. in.

$$\text{The value of } \theta \text{ (Fig. 9)} = 90 - 60 = 30°$$

$$f_n = \frac{W}{A}\cos^2\theta = \frac{5}{1} \times \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{15}{4} = 3{\cdot}75 \text{ tons/sq. in.}$$

$$f_s = \frac{W}{2A}\sin 2\theta = \frac{5}{2} \times \frac{\sqrt{3}}{2} = \frac{8{\cdot}66}{4} = 2{\cdot}165 \text{ tons/sq. in.}$$

Neglecting friction, the maximum shear stress occurs when $\theta = 45°$. Then

$$f_{s\,(max)} = \frac{5}{2} \times 1$$

$$= 2{\cdot}5 \text{ tons/sq. in.}$$

16. **Normal and Shear Stresses** on an oblique section of a bar subjected to **Two Perpendicular Direct Loads.**

The bar shown in Fig. 10 carries perpendicular direct "tensile" stresses of magnitude f_x and f_y. CD is an oblique section. At C and D draw CE and DB perpendicular to the axis of the bar. Then W_x is the load acting on the face CB due to the

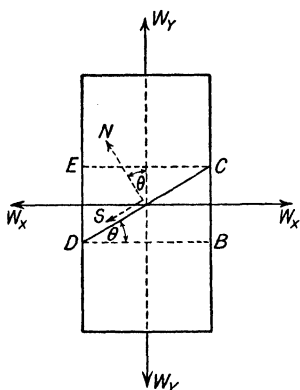
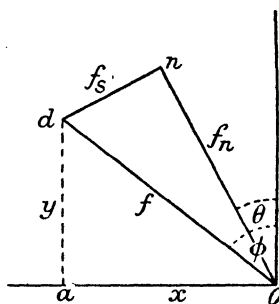



FIG. 10 FIG. 10A

stress f_x and W_y is the load acting on the face EC due to the stress f_y.

If the depth of the bar perpendicular to the plane of the page is unity,

$$\text{then } W_x = f_x \times CB$$

$$\text{and } W_y = f_y \times EC$$

If N and S are the normal and tangential forces on the section at CD and f_n and f_s the corresponding stresses.

$$N = W_y \cos\theta + W_x \sin\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$S = -W_y \sin\theta + W_x \cos\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

but $N = f_n \times CD$ and $S = f_s \times CD$.

$\therefore$ from (1)

$$f_n \times CD = f_y \cos\theta \times EC + f_x \sin\theta \times CB$$

$$f_n = f_y \cos\theta \frac{EC}{CD} + f_x \sin\theta \frac{CB}{CD}$$

$$= f_y \cos\theta \,.\, \cos\theta + f_x \sin\theta \sin\theta$$

$$f_n = f_y \cos^2\theta + f_x \sin^2\theta \qquad . \quad . \quad . \quad . \quad (3)$$

And from (2)

$$f_s \times CD = -f_y \sin\theta \times EC + f_x \times \cos\theta \times CB$$

$$f_s = -f_y \sin\theta \times \frac{EC}{CD} + f_x \times \cos\theta \frac{CB}{CD}$$

$$= -f_y \sin\theta \cos\theta + f_x \times \cos\theta \sin\theta$$

$$f_s = (f_x - f_y) \sin\theta \cos\theta \qquad . \quad . \quad . \quad (4)$$

From (4) $f_s = \dfrac{(f_x - f_y)}{2} \sin 2\theta$

hence f_s will have its maximum value when $\theta = 45°$, and the maximum value is given by

$$f_s = \frac{f_x - f_y}{2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The value of the normal tensile stress f_n corresponding to this maximum shear stress is—

$$f_n = f_y \times \left(\frac{1}{\sqrt{2}}\right)^2 + f_x \times \left(\frac{1}{\sqrt{2}}\right)^2$$

or $$f_n = \frac{f_y + f_x}{2} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let f be the resultant stress of f_n and f_s (3) and (4), then $f = \sqrt{f_n^2 + f_s^2}$

$$= \sqrt{(f_y \cos^2\theta + f_x \sin^2\theta)^2 + (f_x - f_y)^2 \sin^2\theta \cos^2\theta}$$

$$= \sqrt{f_y^2 \cos^4\theta + f_x^2 \sin^4\theta + 2 f_y f_x \sin^2\theta \cos^2\theta + f_y^2 \sin^2\theta \cos^2\theta + f_x^2 \sin^2\theta \cos^2\theta - 2 f_x f_y \sin^2\theta \cos^2\theta}$$

$$= \sqrt{f_y^2 \cos^2\theta (\cos^2\theta + \sin^2\theta) + f_x^2 \sin^2\theta (\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{f_y^2 \cos^2\theta + f_x^2 \sin^2\theta} \qquad . \quad . \quad . \quad . \quad . \quad (7)$$

If f is inclined at an angle α to CD

$$\text{then } \tan \alpha = \frac{f_n}{f_s} = \frac{f_y \cos^2 \theta + f_x \sin^2 \theta}{(f_x - f_y) \sin \theta \cos \theta}$$

$$\text{or } \tan \alpha = \frac{f_x \tan \theta + f_y \cot \theta}{f_x - f_y} \quad . \quad . \quad . \quad (8)$$

Let β be the angle which the resultant f makes with the normal,

$$\text{then } \tan \beta = \frac{f_x - f_y}{f_x \tan \theta + f_y \cot \theta} \quad . \quad . \quad . \quad (9)$$

In order to find the plane which gives the resultant the greatest obliquity, or inclination to the normal, the following procedure may be adopted.

β is a maximum when α is a minimum, that is when $\tan \alpha$ is a minimum, and this occurs when $\frac{d\,(\tan \alpha)}{d\theta} = 0$

$$\text{or} \qquad \frac{1}{f_x - f_y}\left(\frac{f_x}{\cos^2 \theta} - \frac{f_y}{\sin^2 \theta}\right) = 0$$

$$\therefore \tan^2 \theta = \frac{f_y}{f_x}$$

$$\text{and } \tan \theta = \sqrt{\frac{f_y}{f_x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The positive value of which makes $\frac{d^2\,(\tan \alpha)}{d\theta^2}$ positive and hence gives the inclination of the plane which makes α a minimum or β a maximum.

EXAMPLE 2.

A body is subjected to tensile stresses of 2 tons/sq. in. and 4 tons/sq. in. in directions at right angles to one another. Estimate the value of the normal, tangential and resultant stress across a plane inclined 40° to the direction of the 4-ton stress.

Let f_x be the 4-ton stress and f_y the 2-ton stress.

$$\begin{aligned} \text{Then } f_n &= f_y \cos^2 \theta + f_x \sin^2 \theta \\ &= 2 \times (0{\cdot}766)^2 + 4 \times (0{\cdot}6428)^2 \\ &= 1{\cdot}18 + 1{\cdot}66 \\ &= 2{\cdot}84 \text{ tons/sq. in.} \end{aligned}$$

$$\text{and } f_s = \frac{(f_x - f_y)}{2} \sin 2\theta$$

$$= \frac{4 - 2}{2} \times 0{\cdot}9848$$

$$= 0{\cdot}9848 \text{ tons/sq. in.}$$

The resultant stress f

$$= \sqrt{f_n^2 + f_s^2}$$

$$= \sqrt{(2{\cdot}84)^2 + (0{\cdot}9848)^2}$$

$$= 3{\cdot}00 \text{ tons/sq. in.}$$

Unlike Stresses. If f_y, in Fig. 10, acts in the opposite direction, we then have f_x tensile and f_y compressive. The value of f_n and f_s can be found by using the method just adopted, but a quicker result can be arrived at by looking on compressive stresses as negative stresses, and making use of the equations just obtained.

If f_n = resultant *tensile* normal stress on CD

and f_s = resultant shear stress on CD

$$f_n = f_x \sin^2 \theta + (-f_y \cos^2 \theta) = f_x \sin^2 \theta - f_y \cos^2 \theta$$

$$f_s = \frac{f_x - (-f_y)}{2} \sin 2\theta = \frac{f_x + f_y}{2} \sin 2\theta$$

The maximum value of f_s occurs when $\theta = 45°$ and we have the maximum value given by

$$f_s = \frac{f_x + f_y}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Referring to Fig. 10, we see that it is possible to have a load W_z acting in a plane perpendicular to CB and EC, thus giving a stress f_z perpendicular to the plane of the paper.

The greatest shearing stress *in the material* will be half the difference between the greatest and the least stress, due notice being taken of the sign of the smallest stress when it differs from that of the largest stress. In the majority of cases dealt with the value of f_z is zero, then, if f_x is greater than f_y and of the same sign, the greatest shear stress *in the material* is

$$\frac{f_x + 0}{2} = \tfrac{1}{2} f_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

When f_x and f_y are of different signs the greatest shear stress in the material is given by (11).

EXAMPLE 3.

A body is subjected to a stress of 3 tons per sq. in. compression, and a tensile stress of 5 tons per sq. in. in a direction perpendicular to the 3-ton stress. Estimate the magnitude of the maximum shear stress in the body and the value and character of the resultant stress across a section inclined at 35° to the 5-ton stress.

$f_x = 5$ tons/sq. in. tension, $f_y = 3$ tons/sq. in. compression

$\theta = 35°$

$$\therefore f_n = f_x \sin^2\theta - f_y \cos^2\theta = 5 \times (0\cdot5736)^2 - 3 \times (0\cdot8192)^2$$

$$= 0\cdot3665 \text{ tons/sq. in. (compression)}$$

$$f_s = \frac{f_x + f_y}{2} \sin 2\theta = \frac{5+3}{2} \times 0\cdot9397$$

$$= 3\cdot7588 \text{ tons/sq.in.}$$

The resultant stress $\sqrt{f_n^2 + f_s^2} = \sqrt{(-0\cdot3665)^2 + (3\cdot758)^2}$

$= 3\cdot78$ tons/sq. in. (compression).

The maximum shear stress is given by

$$f_{s\,(max)} = \frac{f_x + f_y}{2} = \frac{5+3}{2}$$

$$= 4 \text{ tons/sq. in.}$$

17. **The Ellipse of Stress.** In par. 16 let ϕ be the inclination of f to the direction of W_y, then in Fig. 10 $fDC \sin\phi = W_x = f_x CB$, and $fDC \cos\phi = W_y = f_y EC$.

$$\therefore f \sin\phi = f_x \frac{CB}{DC} = f_x \sin\theta = x \quad \text{(Fig. 10A)}. \qquad (1)$$

$$\text{and } f \cos\phi = f_y \frac{EC}{DC} = f_y \cos\theta = y \qquad (2)$$

$$\therefore \frac{x^2}{f_x^2} + \frac{y^2}{f_y^2} = 1$$

which is the equation of an ellipse, whose axes are equal to $2f_x$ and $2f_y$, and this ellipse is the locus of d.

Such an ellipse is called an *Ellipse of Stress.*

Thus to obtain f graphically the following construction may be used. With centre O, Fig. 11, draw concentric circles of

radii f_x and f_y, and set off OX and OY to represent the direction of each stress. Draw ss at the required inclination, and through O draw Om perpendicular to ss to cut the circle of radius f_y

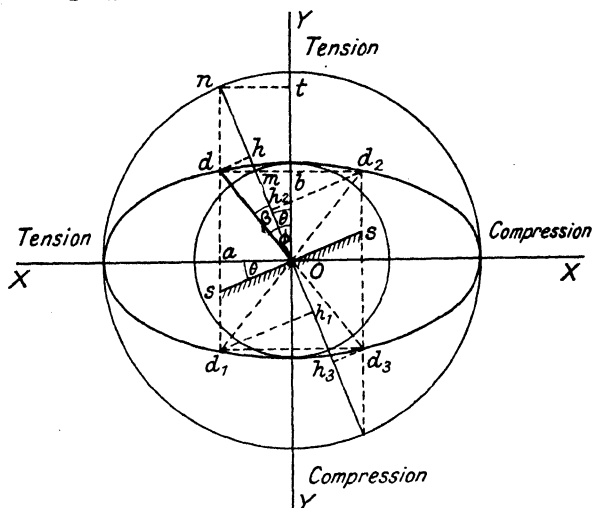



Fig. 11

in m, and the circle of radius f_x in n. Through m draw mb perpendicular to OY, through n draw nd perpendicular to OX and through d draw dh perpendicular to On; then Od represents f to the scale of f_x and f_y, $\widehat{dOb}$ is the inclination of f to the direction Wy and $\widehat{dOh}$ the obliquity, also oh and dh represent the normal and shear stress on ss respectively.

The above may be seen by reference to Fig. 10A or in Fig. 11.

$$\begin{aligned} nm &= f_x - f_y \\ dm &= nm \sin\theta \\ &= (f_x - f_y)\sin\theta \\ dh &= dm\cos\theta \\ &= (f_x - f_y)\sin\theta\cos\theta \\ &= \frac{f_x - f_y}{2}\sin 2\theta \\ &= f_s \qquad \qquad (3) \end{aligned}$$

also $\quad Oh = Om + mh$

$$= f_y + (f_x - f_y) \sin^2 \theta$$
$$= f_y (1 - \sin^2 \theta) + f_x \sin^2 \theta$$
$$= f_y \cos^2 \theta + f_x \sin^2 \theta$$
$$= f_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and $\quad Od^2 = Oh^2 + dh^2$
$$= f_n^2 + f_s^2$$
$\therefore \quad Od = f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$

In Fig. 11 the tension axes are represented above the plane and the compression axes below the plane. The four possible arrangements of a pair of stresses with the accompanying solutions are given. The following rules should be observed in all cases—

(*a*) The major and minor axes should correspond with the axes of major and minor stress respectively.

(*b*) Project the major and minor radii perpendicular to the axes of major and minor stress respectively.

(*c*) If a stress is compressive then produce the normal below the plane *SS*, and project from where the normal produced cuts the circle concerned.

(*d*) If the resultant and normal stresses fall above *SS* they are tensile, and if they fall below *SS* they are compressive.

Examples 2 and 3 should also be solved by the Ellipse of Stress.

18. **Complementary Shear Stresses Required for Equilibrium.** In Fig. 12 let *OABC* be a rectangular block of unit thickness. Suppose a shear stress equal in magnitude to f_{s1} acts on the faces *CB* and *OA*, these being taken perpendicular to the plane of the paper. The forces caused by these stresses form a couple, and for equilibrium of the block, they must be balanced by a couple. Let f_{s2} be the magnitude of the intensity of the forces forming the balancing couple and acting on the faces *OC* and *AB*. Then f_{s2} have the faces *OC* and *AB* in a state of shear.

Fig. 12

Taking moments about *O* for equilibrium of the block we have
$f_{s2} \times$ area of face $AB \times OA = f_{s1} \times$ area of face $CB \times OC$,

$$\text{or } f_{s2} \times a \times 1 \times b = f_{s1} \times b \times 1 \times a$$
$$\therefore f_{s2} = f_{s1}.$$

Thus it is seen that a shear is automatically accompanied by a shear of equal intensity, but opposite turning moment in a direction perpendicular to that of the original shear.

19. Simple Shear Introducing Normal Direct Stresses. Let shear stresses of magnitude f_s act on the faces AB and CD of the cube $ABCD$ of side x, Fig. 13. By the previous paragraph these are accompanied by a shear of equal intensity on the faces AD and BC.

Consider the equilibrium of the portion ABC, Fig. 13*a*. The total force on each of the faces AB and BC

$$= f_s\, x^2.$$

Let f_n = the intensity of the normal force acting in the direction of the diagonal DB for equilibrium of the forces on the faces AB and BC.

Total force on face $AC = \sqrt{2}\, x^2 f_n$.

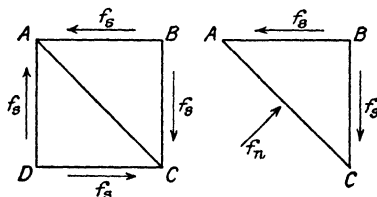



FIG. 13 FIG. 13A

Resolving horizontally (or vertically),

$$\sqrt{2}\, x^2 f_n \times \frac{1}{\sqrt{2}} = x^2 f_s$$

$$\therefore \quad f_n = f_s,$$

and obviously f_n is a compressive stress. If we consider the equilibrium of the portion DAB, it can be shown that the intensity of the *tensile* stress on the face DB is also equal in magnitude to f_s.

Hence, two shear stresses on planes perpendicular to each other are equivalent to tensile and compressive stresses of intensity equal to that of the shear stress, on planes inclined at 45° to the shear stress.

0. **Poisson's Ratio.** When the material of a body is subjected to direct tension or compression a change of length occurs in the direction of the applied force. There is also a change in dimensions in all directions perpendicular to the

applied force and the strains in these directions are spoken of as *lateral strains*

$$\frac{\text{The lateral strain}}{\text{The longitudinal strain}} = \text{constant} = \frac{1}{m}$$

The value of m is found experimentally to lie between the values 3 and 4 for most metals.

The reciprocal of $m = \sigma$ is called *Poisson's Ratio.*

If f_t = longitudinal stress

and e_t = ,, strain

then
$$\text{Lateral strain} = \frac{e_t}{m} = \frac{f_t}{mE}$$

21. **Relations between** E, C, **and** K. A cube of material acted on by a simple shear will be strained as shown in Fig. 14.

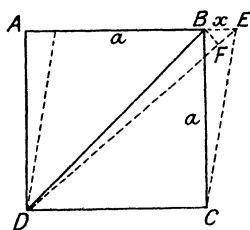



Fig. 14

The movement x of the corner B of the cube is exceedingly small, and the shear strain $= \frac{BE}{BC} = \frac{f_s}{C}$.

From B drop BF perpendicular to DE. Then DF is approximately equal to DB. The diagonal DB is elongated by an amount $FE = BE \cos 45° = \frac{BE}{\sqrt{2}}$, and DB is equal to $BC\sqrt{2}$. Hence the strain in the diagonal DB is given by

$$\frac{FE}{DB} = \frac{BE}{\sqrt{2}} \times \frac{1}{BC\sqrt{2}} = \frac{1}{2}\frac{BE}{BC} = \frac{1}{2}\frac{f_s}{C}$$

Thus it is seen that the strain, in the diagonal of a cube subjected to simple shear, is half the amount of the shear strain. Now in par. 19 it was shown that $f_s = f_n$, hence the strain in the diagonal may be written $\frac{1}{2}\frac{f_n}{C}$ where f_n is the intensity of the equal and opposite direct stresses on planes in the direction of the diagonals.

If f_n acted alone, in the direction of the diagonal DB the

resultant strain would be $\frac{f_n}{E}$, but there is an equal and opposite direct stress in the direction of the diagonal AC, and by par. 19 the strain caused by it in the direction $DB = \frac{1}{m}\frac{f_n}{E}$.

Therefore the total strain in the direction of the diagonal DB

$$= \frac{f_n}{E} + \frac{1}{m}\left(\frac{f_n}{E}\right)$$

$$= \frac{f_n}{E}\left(1 + \frac{1}{m}\right)$$

since the stress in the direction of the diagonal AC tends to increase the strain in the direction of the diagonal DB.

But the strain in the direction $DB = \frac{1}{2}\frac{f_n}{C}$

$$\therefore \frac{1}{2}\frac{f_n}{C} = \frac{f_n}{E}\left(1 + \frac{1}{m}\right)$$

$$\frac{1}{2C} = \frac{1}{E}\left(1 + \frac{1}{m}\right)$$

$$\therefore E = 2C\left(1 + \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad (1)$$

Now consider a cube under an equal compressive load on all its faces such that the direct stress is f_n. The bulk or volume strain $= \frac{f_n}{K}$ where K is the bulk or volume modulus.

The forces acting parallel to an edge of the cube cause it to be shortened so that the strain is $\frac{f_n}{E}$, but the forces acting perpendicularly to the edge considered cause it to be strained by the amount $\frac{1}{m}\frac{f_n}{E} + \frac{1}{m}\frac{f_n}{E} = \frac{2}{m}\frac{f_n}{E}$

Thus the total strain along the edge $= \frac{f_n}{E} - \frac{2}{m}\frac{f_n}{E}$

$$= \frac{f_n}{E}\left(1 - \frac{2}{m}\right)$$

Now the volumetric strain of a cube acted on by forces as stated was shown in par. 7 (*d*) to be equal to three times the linear strain of the edge.

Therefore the volumetric strain is given by

$$\frac{3f_n}{E}\left(1-\frac{2}{m}\right)$$

$$\therefore \frac{f_n}{K}=\frac{3f_n}{E}\left(1-\frac{2}{m}\right)$$

$$\frac{1}{K}=\frac{3}{E}\left(1-\frac{2}{m}\right)$$

$$E=3K\left(1-\frac{2}{m}\right) \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the value of E found in (1),

$$2C\left(1+\frac{1}{m}\right)=3K\left(1-\frac{2}{m}\right)$$

$$\therefore \frac{1}{m}=\frac{3K-2C}{6K+2C} \quad . \quad . \quad . \quad . \quad (3)$$

Also from (1) $\quad \frac{1}{m}=\frac{E-2C}{2C}$

$$\therefore \frac{E-2C}{2C}=\frac{3K-2C}{6K+2C}$$

or

$$E=\frac{9CK}{3K+C} \quad . \quad . \quad . \quad . \quad (4)$$

EXAMPLE 4.

A cylindrical bar of steel 2 in. diameter and 12 in. long, when subjected to an axial tensile load is found to have a modulus of elasticity of 29,600,000 lb./sq. in. and a ratio of longitudinal to lateral strain of 3·4 : 1. Calculate its modulus of rigidity, and find also the change of volume produced in this bar if submitted to a fluid stress of 5 tons/sq. in. (Lond. Univ., 1923.)

$$\frac{\text{Lateral strain}}{\text{Longitudinal strain}}=\frac{1}{m}=\frac{1}{3{\cdot}4}$$

$$E=3K\left(1-\frac{2}{m}\right)$$

$$\therefore K=\frac{E}{3\left(1-\frac{2}{m}\right)}=\frac{E}{3\left(1-\frac{2}{3{\cdot}4}\right)}$$

$$= \frac{E}{1 \cdot 239}$$

$$\text{and } E = \frac{9CK}{3K + C}$$

$$E = \frac{9C \dfrac{E}{1 \cdot 239}}{\dfrac{3E}{1 \cdot 239} + C}$$

$$\frac{3E^2}{1 \cdot 239} + EC = \frac{9}{1 \cdot 239} EC$$

$$C = \frac{2 \cdot 42}{6 \cdot 26} E$$

$$\therefore C = 11{,}450{,}000 \text{ lb./sq. in.}$$

$$\text{Volumetric strain} = \frac{f_n}{K} = \frac{5 \times 2240 \times 1 \cdot 239}{29{,}600{,}000}$$

$$\therefore \text{Change in volume of bar} = \frac{5 \times 2240 \times 1 \cdot 239}{29{,}600{,}000} \times \pi \times 1 \times 12$$

$$= 0 \cdot 01800 \text{ cub. in.}$$

22. **Principal Stresses and Principal Planes.** At any point within a body, no matter how complex the state of stress may be, it will be found that there exist three mutually perpendicular planes on each of which the resultant stress is a normal stress. These mutually perpendicular planes are called *Principal Planes*, and the resultant normal stresses acting on them are called *Principal Stresses*. In the majority of cases met with in the study of the strength of materials one of the principal stresses is zero. We are then only concerned with a state of stress in two dimensions. It will be found that one of the principal stresses in this case is the greatest direct stress to which the material is subjected, and the other principal stress is the least direct stress to which the material is subjected. In par. 15 the stress f on the section DB is a principal stress, and so also are the stresses f_x and f_y in par. 16. In par. 19 it was shown that a simple shear acting on a body introduces two principal stresses f_n on planes inclined at 45° to the directions of the original shear, and equal in magnitude to the shear.

We will now consider a more general case of complex stress as represented by Fig. 15, in which *ABCD* represents a cube of side a. The faces at *BC* and *AD* are subjected to direct tensile stress of magnitude f_y and the faces at *CD* and *AB* are each subjected to direct tensile stress of magnitude f_x. Each of the above faces is also subjected to a shear stress of magnitude f_s. On planes parallel to the plane of the paper no stress is assumed to act.

Let one of the principal planes *CE* make an angle θ with *AD*. The normal stress f_n on *CE* is then a principal stress, and the

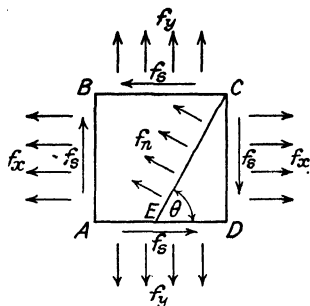
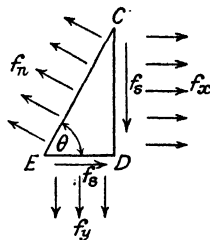
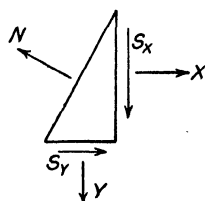



FIG. 15 FIG. 15A FIG. 15B

force due to this stress balances the forces acting on the triangular piece *CED*.

$$\text{Area of face at } CD = a^2$$

$$\text{,, \quad ,, \quad ,, } CE = \frac{a^2}{\sin\theta}$$

$$\text{,, \quad ,, \quad ,, } ED = \frac{a^2}{\tan\theta}$$

$$\text{Total direct force on face } CD = f_x a^2 = X$$

$$\text{,, \quad shear \quad ,, \quad ,, } CD = f_s a^2 = S_x$$

$$\text{,, \quad direct \quad ,, \quad ,, } ED = f_y \frac{a^2}{\tan\theta} = Y$$

$$\text{,, \quad shear \quad ,, \quad ,, } ED = f_s \frac{a^2}{\tan\theta} = S_y$$

$$\text{,, \quad direct \quad ,, \quad ,, } EC = f_n \frac{a^2}{\sin\theta} = N$$

Resolving horizontally,

$$N \sin \theta = X + S_y$$

$$f_n \frac{a^2 \sin \theta}{\sin \theta} = f_x a^2 + f_s \frac{a^2}{\tan \theta}$$

$$\text{or } f_s = (f_n - f_x) \tan \theta \quad . \quad . \quad (1)$$

Resolving vertically,

$$N \cos \theta = S_x + Y$$

$$f_n \frac{a^2 \cdot \cos \theta}{\sin \theta} = f_s a^2 + f_y \frac{a^2}{\tan \theta}$$

$$\therefore f_s = \frac{f_n - f_y}{\tan \theta} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2) by subtraction,

$$f_s (\cot \theta - \tan \theta) = f_y - f_x$$

$$\text{or } 2f_s \cot 2\theta = f_y - f_x$$

$$\therefore \tan 2\theta = \frac{2f_s}{f_y - f_x} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

an equation giving two values of θ, differing by 90°, in terms of the original stresses.

Also from (1) and (2) by multiplication,

$$f_s^2 = (f_n - f_x)(f_n - f_y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\text{A})$$

or

$$f_n^2 - f_n (f_x + f_y) + f_x f_y - f_s^2 = 0$$

$$f_n = \frac{(f_x + f_y) \pm \sqrt{(f_x + f_y)^2 - 4(f_x f_y - f_s^2)}}{2}$$

or

$$f_n = \tfrac{1}{2}\{(f_x + f_y) \pm \sqrt{(f_x - f_y)^2 + 4f_s^2}\} \quad . \quad . \quad . \quad (4)$$

The two values of f_n obtained from this equation give the two principal stresses, the maximum principal stress being obtained when the plus sign is taken, and the minimum principal stress when the minus sign is taken. It is obvious from the equation that the sign of the maximum principal stress will be the same as f_x and f_y. To get the sign of the minimum principal stress we must note whether $(f_x + f_y)$ or $\sqrt{(f_x - f_y)^2 + 4f_s^2}$ is the greater. Squaring each, we observe that the first will be the greater if $f_x f_y$ is greater than f_s^2. Hence the minimum principal stress has the same sign as f_x and f_y if $f_x f_y$ is greater than f_s^2.

In order to find f_n for any modification of the original stresses it is only necessary to reverse the sign of a stress when it is compressive, or eliminate the stress if it is zero.

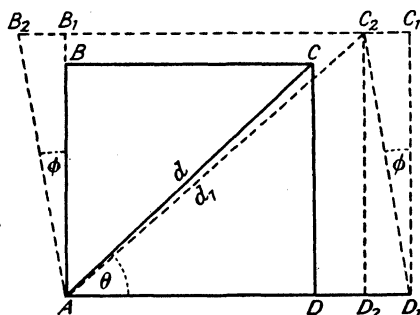



FIG. 16

23. **Principal Strains.** The maximum and minimum direct strains in a material, subjected to complex stress, are called the *Principal Strains.* These strains act in the directions of the principal stresses. Let $ABCD$, Fig. 16, be a block acted on by direct stresses f_x and f_y, and a shear stress f_s as in par. 22.

Let e_x = the total strain in direction A to $D = \dfrac{DD_1}{AD} = \dfrac{f_x}{E} - \dfrac{f_y}{mE}$

Let e_y = ,, ,, ,, ,, A to $B = \dfrac{BB_1}{AB} = \dfrac{f_y}{E} - \dfrac{f_x}{mE}$

Owing to the shear stress acting on the block, its final shape will be $AB_2C_2D_1$. The direct strain in the direction A to C is

$$(AC_2 - AC)\frac{1}{AC} = e_d$$

In the following argument squares and higher powers of strains will be neglected since a strain is exceedingly small.

$$AC_2^2 = AD_2^2 + C_2D_2^2 = (AD_1 - C_1C_2)^2 + AB_1^2$$
$$= \{AD(1 + e_x) - \phi AB(1 + e_y)\}^2 + AB^2(1 + e_y)^2$$
$$= AD^2(1 + 2e_x) - 2\phi AB\,AD + AB^2(1 + 2e_y) \quad . \quad (1)$$
$$AC_2^2 = 2AD^2e_x - 2\phi\,AB\,AD + 2AB^2e_y + AC^2 \quad . \quad . \quad (2)$$

and $AC_2^2 = AC^2(1 + 2e_d)$.

$$\therefore 2AC^2e_d = 2AD^2e_x + 2AB^2e_y - 2AB\,AD\phi$$
$$e_d = \left(\frac{AD}{AC}\right)^2 e_x + \left(\frac{AB}{AC}\right)^2 e_y - \frac{AB}{AC}\cdot\frac{AD}{AC}\,\phi$$
$$= e_x\cos^2\theta + e_y\sin^2\theta - \phi\sin\theta\cos\theta$$

or $$e_d = \tfrac{1}{2}\{e_x + e_y + \cos 2\theta\,(e_x - e_y) - \phi\sin 2\theta\} \quad . \quad . \quad (3)$$

The maximum and minimum values of (3) are given when

$$\frac{d(e_d)}{d\theta} = 0. \qquad \text{i.e. } \tan 2\theta = \frac{-\phi}{e_x - e_y} \quad . \quad . \quad . \quad . \quad (4)$$

$$= \frac{\frac{-f_s}{C}}{\frac{f_x}{E} - \frac{f_y}{mE} - \frac{f_y}{E} + \frac{f_x}{mE}} = \frac{-f_s}{f_x - f_y} \cdot \frac{E}{C\left(1 + \frac{1}{m}\right)}$$

Hence $\tan 2\theta_{max} = \frac{2f_s}{f_y - f_x}$ . . . . . . (5)

Thus the maximum and minimum direct strains occur in directions given by (5), and by reference to par. 22 (3), it is seen that the directions of the principal stresses are also given by (5). Hence the principal strains and principal stresses act in the same directions. The two values of principal strain are given by $e_d = \frac{1}{2}\{(e_x + e_y) \pm \sqrt{(e_x - e_y)^2 + \phi^2}\}$ . . . (6)
or from the solution of

$$(e_d - e_x)(e_d - e_y) = \phi_1^2 \quad . \quad . \quad . \quad (7)$$

and $\tan \theta = \frac{e_d - e_y}{\phi_1}$ . . . (8) where $\phi_1, = \frac{1}{2} \phi$

23A. **Ellipse of Strain.** The rectangular plate $DECB$, Fig. 16A, is acted on by stresses such that e_1 and e_2 are the principal strains. The rectangle takes up the shape $D_1E_1C_1B_1$ and the strain in OC due to e_1 and e_2 is given by $= \frac{KC_1}{OC} = \frac{KM + MC_1}{OC}$

$$= \frac{CH \cos\theta + C_1H \sin\theta}{OC} = \frac{\frac{1}{2}ECe_1 \cos\theta + \frac{1}{2}CBe_2 \sin\theta}{\frac{1}{2}DC}$$

$$= e_1 \cos^2\theta + e_2 \sin^2\theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The strain in a perpendicular direction is given by $\frac{CK}{OC}$

$$= \frac{CF - KF}{OC} = \frac{CH \sin\theta - C_1H \cos\theta}{OC}$$

$$= \frac{\frac{1}{2}ECe_1 \sin\theta - \frac{1}{2}CBe_2 \cos\theta}{\frac{1}{2}DC}$$

$$= \frac{e_1 - e_2}{2} \sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In Fig. 16B if oh and hd represent the strains along and perpendicular to OC, then Od is the resultant strain. It is easily seen that if $Oa = x = e_1 \cos\theta$ and $ad = y = e_2 \sin\theta$, then $\frac{x_2}{e_1^2} + \frac{y^2}{e_2^2} = 1$. This is the equation of an ellipse of major axis $2e_1$ and minor axis $2e_2$ and is the locus of d. Such an ellipse is called an *Ellipse of Strain.* Draw two concentric circles of radii e_1

and e_2 and draw *On* at an angle θ to the direction of e_1, project *na* and *mb* as shown and join *Od*. *Oh* is the strain in the direction θ since by resolving it is found equal to (1), and similarly the strain in a perpendicular direction is given by *hd* since it has the value $\frac{1}{2}(e_1 - e_2)\sin 2\theta$, *Od* is therefore the resultant strain.

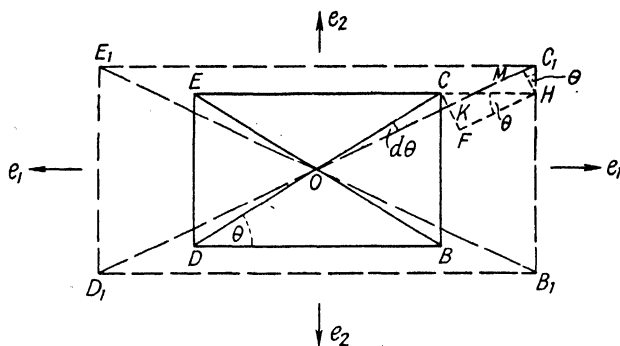



Fig. 16A

Since each diagonal of the plate, Fig. 16A, rotates through the angle $d\theta$ the shear strain in the material is given by

$$2d\theta = 2\frac{CK}{OC} = (e_1 - e_2)\sin 2\theta \quad . \quad . \quad . \quad . \quad . \quad (3)$$

i.e. twice the magnitude of *hd* in Fig. 16B.

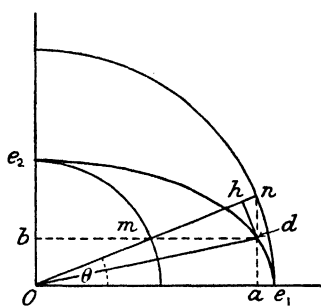



Fig. 16B

The *maximum* shear strain is given by $(e_1 - e_2)$ and occurs on planes making 45° with the directions of the principal strains.

By substituting the values of e_1 and e_2 as given by (6), par. 23, the maximum shear strain is found in terms of e_x, e_y and ϕ, and is given by $\sqrt{(e_x - e_y)^2 + \phi^2}$.

24. **Single Direct Stress to Produce Given Maximum Strain.** In the previous paragraph the maximum strain is produced by a complex system of stressing. Suppose f to be the magnitude of the single direct stress acting in the material and causing a strain of magnitude given by (6), par. 23.

The strain due to $f = \dfrac{f}{E} = e_d$

Now $e_d = \frac{1}{2}\{(e_x + e_y) + \sqrt{(e_x - e_y)^2 + \phi^2}\}$

$$\therefore \frac{f}{E} = \frac{1}{2}\left\{\left(\frac{f_x}{E} - \frac{f_y}{mE} + \frac{f_y}{E} - \frac{f_x}{mE}\right) + \sqrt{\left(\frac{f_x}{E} - \frac{f_y}{mE} - \frac{f_y}{E} + \frac{f_x}{mE}\right)^2 + \frac{f_s^2}{C^2}}\right\}$$

$$\text{or } f = \frac{1}{2}\left\{(f_x + f_y)\left(1 - \frac{1}{m}\right) + \left(1 + \frac{1}{m}\right)\sqrt{(f_x - f_y)^2 + 4f_s^2}\right\}$$

25. **Assumptions made in Dealing with Complex Stresses.** It should be observed from the treatment of complex stresses that the following important assumptions have been made—

(1) That the material is isotropic; that is, it possesses the same elastic properties in all directions.

(2) That the principal stresses are linear functions of the principal strains.

26. **Important Modification of General Equation for Principal Stresses.** An important case of the modification of the general equation given by (4), par. 22, is that where f_y is zero. The state of stress then is that existing in a shaft subjected to bending and torsion. The direction of f_x must be changed, however, for the other side of the shaft.

The principal stresses for one side of the shaft are given by

$$f_n = \frac{1}{2}\{f_x \pm \sqrt{f_x^2 + 4f_s^2}\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The maximum principal stress will have the same sign as f_x, but the minimum principal stress will be negative, i.e. compressive, since $\sqrt{f_x^2 + 4f_s^2}$ is greater than f_x.

The directions of the principal stresses will be given by

$$\tan \theta = \frac{f_n}{f_s} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In par. 16 it was shown that the maximum shear stress, due to two principal tensile stresses, was equal to half the difference of the stresses. Therefore the maximum shear stress in a shaft subjected to bending and torsion is given by

$$f_{s\,max} = \frac{\frac{1}{2}\{f_x + \sqrt{f_x^2 + 4f_s^2}\} - \frac{1}{2}\{f_x - \sqrt{f_x^2 + 4f_s^2}\}}{2}$$

$$= \frac{\sqrt{f_x^2 + 4f_s^2}}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

27. **Modified Values of Young's Modulus.** The value of E was defined in Chap. I, par. 8, as $\frac{f_t}{e_t}$ or $\frac{f_c}{e_c}$, and in each case only one principal stress was acting on the material. If, however,

more than one principal stress acts, then the resultant strain will be altered and hence the value of E will be modified. The following example will serve to make the point clear—

EXAMPLE 5.

A solid cylinder is compressed in the direction of its axis, and means are employed to reduce lateral expansion to one-third of what it would be if free to expand. Find the value of the elastic constant for the material, the stress necessary to reduce expansion, and the ratio of the axial strain to that in a cylinder free to expand.

$$e_x = \frac{f_x}{E} - \frac{f_y + f_z}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$e_y = \frac{f_y}{E} - \frac{f_x + f_z}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$e_z = \frac{f_z}{E} - \frac{f_x + f_y}{mE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

f_y will be equal to f_z and hence from (1) we get

$$e_x = \frac{f_x}{E} - \frac{2f_y}{mE} = \text{axial strain}$$

and from (2) $e_y = \dfrac{f_y}{E} - \dfrac{f_x + f_y}{mE}$

For a cylinder free to expand $e_y{}^1 = \dfrac{-f_x}{mE}$ reckoning compressive strain as positive.

$$\therefore \frac{f_y}{E} - \frac{f_x + f_y}{mE} = \frac{-f_x}{3mE}$$

$$3mf_y - 3f_x - 3f_y = -f_x$$

$$f_y = \frac{2}{3} \cdot \frac{f_x}{m - 1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or if $m = 4$ then

$$f_y = \frac{2}{9} f_x$$

From (1) $E\, e_x = f_x - \dfrac{f_y + f_z}{m}$

$$= f_x - \frac{2f_y}{m}$$

$$= f_x - \frac{4}{3m}\left(\frac{f_x}{m - 1}\right)$$

$$= f_x \left(\frac{3m^2 - 3m - 4}{3m\,(m - 1)}\right)$$

$$\therefore \qquad \frac{f_x}{e_x} = \frac{3m\,(m-1)}{3m^2 - 3m - 4}\,E$$

The elastic constant $= \frac{f_x}{e_x}$, under the given conditions, is thus $\frac{3m\,(m-1)}{3m^2 - 3m - 4}$ times that when free expansion is allowed, or taking $m = 4$ the new value of the elastic constant is $\frac{9}{8}$ times the ordinary value.

Also $e_x^1 = \frac{f_x}{E}$ where $e_x^1 =$ strain when expansion is free.

$$\therefore \qquad \frac{e_x}{e_x^1} = \frac{3m^2 - 3m - 4}{3m\,(m-1)}$$

$$= \frac{8}{9} \text{ when } m = 4$$

28. **Strain Energy Due to Complex Stress.** Let the principal stresses, due to a complex system of stress in two dimensions, be f_{n1} and f_{n2}, and the strains in the directions of the principal stresses be e_{d1} and e_{d2}. From the assumptions already made with regard to principal stresses and principal strains the strain energy per unit volume will be given by

$$W = \tfrac{1}{2} f_{n1}\, e_{d1} + \tfrac{1}{2} f_{n2}\, e_{d2}$$

$$\frac{f_{n1}}{2}\left\{\frac{f_{n1}}{E} - \frac{f_{n2}}{mE}\right\} + \frac{f_{n2}}{2}\left\{\frac{f_{n2}}{E} - \frac{f_{n1}}{mE}\right\}$$

$$= \frac{1}{2E}\left\{f_{n1}^2 + f_{n2}^2 - \frac{2f_{n1}\,f_{n2}}{m}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the stress system is similar to that given in par. 22, then substituting the values of f_{n1} and f_{n2} from (4),

$$W = \frac{f_x^2 + f_y^2}{2E} - \frac{f_x f_y}{mE} + \frac{f_s^2}{E}\left(1 + \frac{1}{m}\right)$$

but from par. 21 (1) $\left(1 + \frac{1}{m}\right)\frac{1}{E} = \frac{1}{2C}$

$$\therefore\ W = \frac{f_x^2 + f_y^2}{2E} - \frac{f_x f_y}{mE} + \frac{f_s^2}{2C} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

if f_x and f_y are zero then the strain energy due to shear is given by

$$W_s = \frac{1}{2}\frac{f_s^2}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

29. **Theories of Failure Under Compound Stresses.** Owing to the large number of examples of compound stresses met with in engineering practice, the cause of "failure" or permanent set, under such conditions, has attracted considerable attention. Certain theories, based on the preceding equations, have been advanced to explain the cause of failure, and many of these theories have received considerable experimental investigation. No great uniformity of opinion has been reached, however, and there is still room for a great deal of further experimental investigation.

(*a*) *Maximum Principal Stress Theory.* This theory is usually associated with Rankine, but also received considerable support from other writers. According to this theory permanent set takes place, under a state of complex stress, when the value of the maximum principal stress is equal to that of the elastic limit as found in a simple tensile test.

(*b*) *Maximum Principal Strain Theory.* This theory is associated with St. Venant, and according to it permanent set takes place when the stress corresponding to the maximum principal strain is equal to the elastic limit in simple tension.

(*c*) *Maximum Shear Stress Theory.* According to Guest,* the criterion of failure is the greatest shear stress, and hence failure will depend on the difference between the greatest and least principal stresses reaching a given value.

(*d*) *Strain Energy Theory.* This theory, which has a thermodynamic analogy and a logical basis, is due to Haigh.† He contends that if a body can be brought to a particular state by various methods then the work done in passing from the initial to the final state will be independent of the method employed. A relation between the elastic limit of a material as found in a simple tension test, and that under compound stresses is thus obtained, and Haigh's theory states that when a material is caused to take permanent set (reach the elastic limit) by stresses which increase gradually from zero, then the unital strain energy is independent of the nature of the stresses and is almost constant in value.

* *Phil. Mag.*, July, 1900. † B.A. Reports, 1919 and 1921.

The theory is comparatively new and much experimental verification is required.

EXAMPLES II

1. A short cast-iron pillar of 1 sq. in. cross-section carries an axial compressive load of 10 tons. Calculate the magnitude of the normal and shear stress on a plane inclined at 30° to the axis of the pillar. What is the value of the maximum shear stress in the material ? *Ans.*, 2·5, 4·33 and 5 tons/sq. in.

2. A steel tie-bar is 30 ft. long, and 4 in. × 1 in. in section. It is subjected to an axial pull of 30 tons, and the increase in length is found to be 0·2 in. Find—

(*a*) The intensity of stress on a right cross-section.

(*b*) The intensities of stress, normal and tangential, on a plane section inclined at 60° to the longitudinal axis.

(*c*) The modulus of elasticity for the steel. (A.M.I.Mech.E., 1924.)

Ans., 7·5, 5·63, and 3·25 tons/sq. in., 13,500 tons/sq. in.

3. A round bar 3 in. diameter and 18 ft. long carries a load of 28 tons and extends $\frac{1}{16}$ in. Estimate the maximum shear stress in the bar and the value of the modulus of elasticity. *Ans.*, 1·98 tons/sq. in. 13,700 tons/sq. in.

4. If a body is subjected to tensile stresses of intensity p_1 and p_2 at right angles to one another, show how to find the normal and tangential stresses on a plane face inclined at an angle θ to the line of stress p_1. A steel plate is subjected to tensile stresses of 8 and 5 tons/sq. in. at right angles to one another. Determine the normal and tangential stresses on a plane inclined at 60° to the 8-ton stress. (Lond. Univ., 1919.) *Ans.*, 7·25, 1·299 tons/sq. in.

5. The principal stresses at a point in a piece of steel are 6 tons per sq. in. tensile, 4 tons/sq. in. compressive, and zero. Find the intensity and direction of the stress across a plane the normal of which is inclined 30° to the axis of the 6-ton principal stress, this plane being also perpendicular to the plane of zero stress. (A.M.I.Mech.E., 1926.)

Ans., resultant stress 5·565 tons/sq. in. tensile inclined to plane at 39°.

6. Explain what is meant by the "Ellipse of Stress." If the principal stresses at a point in a material are 8, 4, and 0 tons per sq. in. tensile, find the resultant stress in magnitude and direction on a plane inclined 30° to the principal plane across which the principal stress is 8 tons/sq. in. and perpendicular to that across which the stress is zero. (A.M.I.Mech.E., 1925.)

Ans., 7·2 tons/sq. in. inclined at 76° to plane.

7. Prove the following relations between the important elastic constants—

$$K = \frac{mE}{3\,(m - 2)}; \quad C = \frac{mE}{2\,(m + 1)}.$$

Show that the coefficient of direct elasticity when lateral contraction is prevented in one direction is $E\,\frac{m^2}{m^2 - 1}$.

8. Establish the relations of Young's modulus, the modulus of rigidity, and Poisson's ratio. If the modulus of rigidity for a particular metal is 8 million lb. per sq. in. and the elastic lateral strain (contraction) in tension is 23 per cent of the direct tensile strain, calculate the value of Young's modulus. (A.M.I.Mech.E., 1926.) *Ans.*, $19{\cdot}68 \times 10^6$ lb./sq. in.

9. Prove that the modulus of rigidity is $\frac{2}{5}$ of Young's modulus of elasticity when Poisson's ratio is 4. (Lond. Univ., 1915.)

10. Determine the maximum intensities of direct and shear stress in a shaft in which there is a shear stress of 3·5 tons/sq. in. and a tensile stress of 4 tons/sq. in. *Ans.*, 6·03 and 4·03 tons/sq. in.

11. At a point in the section of a beam there is a shear stress of 3 tons/sq. in. and a tensile stress of 5 tons/sq. in. Find the principal stresses in magnitude and direction at the point. Prove the formula you use. (Lond. Univ., 1914.)
Ans., 6·4 tensile and 1·4 compressive, tons/sq. in. $\theta = 65°$ and 155°.

12. The principal stresses at a point in a member subjected to two-dimensional stress are 8 and 3 tons/sq. in. tensile respectively. Find the plane on which the resultant stress has maximum obliquity. (Lond. Univ., 1921.)
Ans., $\theta = 58·5°$ to direction of 8-ton stress.

13. At a point in the web of a loaded girder the longitudinal tensile stress is 4·8 tons/sq. in. and the shear stress is 2·5 tons/sq. in. Find the magnitude and direction of the principal stresses at the point. What is the magnitude and direction of the greatest shear stress ? (Lond. Univ., 1922.)
Ans., 5·86 tensile and 1·065 compressive tons/sq. in., $\theta = 67°$ and 157°, 3·47 tons/sq. in.

14. Define "Poisson's Ratio." A steel bar 20 ft. long and 3 in. square is elongated by a load of 50,000 lb. If the modulus of elasticity of the material is 30,000,000 lb./sq. in. and Poisson's ratio is $\frac{1}{4}$, find the increase in volume of the bar. (Lond. Univ., 1918.) *Ans.*, 0·2 cu. in.

15. Define "Young's Modulus" and "Poisson's Ratio." A piece of cast iron, 6 in. long by 1 in. square, is in compression under a load of 9 tons. If the modulus of elasticity of the material is 7,500 tons/sq. in. and Poisson's ratio is $\frac{1}{4}$, find the alteration in length if all lateral strain is prevented by the application of uniform lateral external pressure of suitable intensity. (Lond. Univ., 1918.) *Ans.*, 0·006 in.

16. A rectangular block sustains stresses in three directions at right angles to each other of 5 tensile, 4 compressive, and 6 tensile tons/sq. in. respectively. Assuming the value of Poisson's ratio for this material is $\frac{1}{3·5}$, and that the value of E is 13,000 tons/sq. in., determine the strain in each of the three directions and the values of the volume and torsion moduli. (Lond. Univ., 1921.) *Ans.*, 0·000341, 0·000548, 0·00044, 10,111 tons/sq. in., 5055·5 tons/sq.in.

17. Obtain a formula connecting the moduli of elasticity and rigidity and the ratio of longitudinal to lateral strain. A bar of steel $1\frac{1}{2}$ in. diameter is subjected to a tensile load of 10 tons, and the measured extension on an 8 in. length is 0·0034 in., and the change in diameter 0·00019 in. What is the modulus of rigidity of this steel ? (Lond. Univ., 1922.) *Ans.*, 5,150 tons/sq. in.

18. A material is subjected to shearing stress of 5 tons/sq. in., tensile stress of 4 tons/sq. in., and compressive stress, in a direction at right angles to the tensile stress, of 3 tons/sq. in. Determine the direction and magnitude of the greatest and least stresses in the material.
Ans., 6·6 tensile and 5·6 compressive tons/sq. in. $\theta = 62\frac{1}{2}°$ and $152\frac{1}{2}°$.

19. A bar of elastic material is subjected to direct stress in a longitudinal direction, and its strains in the two directions at right angles are reduced to one-half and one-third respectively of those which normally occur in an

ordinary tension member. If $E = 30 \times 10^6$ lb./sq. in. and $m = 4$, what is the value of the elastic constant ? *Ans.*, $33 \cdot 2 \times 10^6$ lb./sq. in.

20. A cubical block of material of unit side is strained by direct and shearing strains as follows: A tensile strain of 7×10^{-6}, and a compressive strain of 6×10^{-6}, on faces perpendicular to those carrying the previous strain. The magnitude of the shear strain being 3×10^{-6}. On faces, perpendicular to those carrying the above strains, there is no strain acting. Determine the principal strains in magnitude and direction, and calculate the magnitude of the single direct stress required to produce the maximum principal strain. Assume $E = 30 \times 10^6$ lb./sq. in.

Ans., $7 \cdot 09 \times 10^{-6}$ tensile and $6 \cdot 09 \times 10^{-6}$ compressive, and 212·7 lb./sq. in.

21. Derive the formula for the bulk modulus of a material in terms of Young's modulus and Poisson's ratio. In a tensile test on a steel tube of external diameter 0·75 in. and bore 0·5 in., an axial load of 0·2 tons produced a stretch on a length of 2 in. of $1 \cdot 2 \times 10^{-4}$ in., and a lateral contraction of the outer diameter of $1 \cdot 31 \times 10^{-5}$ in. Calculate Young's modulus, Poisson's ratio, and the bulk modulus for the material. (Lond. Univ., 1933.)

Ans., 13,600 tons/sq. in.; 0·292; 10,900 tons/sq. in.

22. Show how to use a circular diagram to represent the intensity of stress, and its direction on any plane at a point in the material subject to two given principal stresses, the third one being zero. Draw the diagram for the case when one principal stress is 6 tons/sq. in. tensile, and the other 4 tons/sq. in. tensile. Indicate on your diagram the stress in magnitude and direction on a plane inclined at 30° to that of the greater principal stress. (I.Mech.E., 1936.)

Ans., 4·58 tons/sq. in. inclined at 79° to plane.

23. The loads applied to a piece of material cause a shear stress of 4 tons/sq. in. together with a normal tensile stress on a certain plane. Find the value of this tensile stress if it makes an angle of 30° with the major principal stress. What are the values of the principal stresses? (Lond. Univ., 1937.)

Ans., 4·63 tons/sq. in., 6·93 tons/sq. in. tensile, 2·31 tons/sq. in. compressive.

24. A piece of material is subjected to two mutually perpendicular stresses f_1 tensile and f_2 compressive. Find an expression for the strain energy per unit volume.

If a stress of 8 tons per sq. in. acting alone gives the same value of the strain energy per unit volume as the expression already found, find the value of f_2 when f_1 is 7 tons per sq. in. Poisson's ratio 0·32. (Lond. Univ., 1940.)

Ans., 2·235 tons/sq. in. compressive or 6·72 tons/sq. in. tension.

25. At a certain point in a steel structural element the directions of the principal stresses p_x and p_y are known. Measurements by strain gauges show that there is a tensile strain of 0·00083 in the direction of p_x, and a compressive strain of 0·00052 in the direction of p_y. Find the magnitude of p_x and p_y, stating whether tensile or compressive, and the value of the maximum shear stress, $m = 4$. (A.M.I.Mech.E., 1940.)

Ans., 22,400 tension, 10,000 compressive, 16,200 lb./sq. in.

CHAPTER III

SIMPLE MECHANICAL PROPERTIES OF METALS

30. **Properties of Metals.** Properties associated with metals, and of great importance from an engineering standpoint are elasticity, ductility, brittleness, plasticity, and malleability.

Several of these properties are in opposition, and thus a particular metal cannot possess all of them simultaneously. Mild steel possesses elasticity, copper ductility, cast iron is brittle, whilst lead is plastic, and wrought iron malleable. In a simple tensile test on a material, carried to destruction, sufficient information is obtained to form an opinion of the strength, elasticity and ductility of the material. In design these properties are of premier importance, the strength of the material must be known in order to calculate the dimensions, and the remaining properties are desirable, since, for various reasons, high local stresses sometimes occur, and thus it is essential that a certain amount of elastic deformation should take place in order to relieve these local stresses.

31. **Behaviour of Metals in Tension.** When the ductile metals are subjected to a gradually increasing tensile load, the tensile stress is proportional to the tensile strain until a certain point is reached, after which proportionality no longer holds, and slightly before reaching this stage it is found that the elasticity of the material has broken down. The elongation so far has been exceedingly small, and very delicate measuring devices called extensometers are required in order to measure the small changes in length.

If the load be increased a small amount, it will be found that the material " flows or yields " a large amount, and the elongation can be measured with a scale and a pair of dividers. The material is now in a semi-plastic state, and further applications of load cause extensions which increase with time. Each application of load causes the strains to increase more and more rapidly with the stresses, until, finally, a value of load is reached at which the material stretches locally and a " waist " is formed. At this stage, owing to the decrease in cross-section, a smaller load than that at which the waist formed is required

to continue the elongation. With the continuance of elongation the cross-section becomes smaller and smaller. Hence the load necessary will be gradually reduced until fracture occurs.

Fig. 17 represents a stress-strain diagram for a ductile material tested, in tension, to destruction. That portion of the test up to the point at which yield occurs is shown plotted to a larger scale. The stresses are all calculated using the original cross-sectional area of the material, and are called *Nominal*

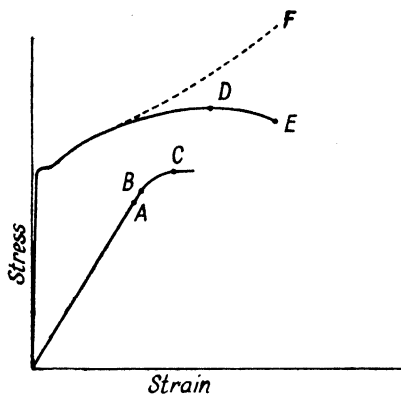



FIG. 17

Stresses, since the actual cross-sectional area does not remain constant. If the actual cross-sectional area was calculated for each load, then the "actual stress" diagram is that shown dotted in Fig. 17. In the nominal stress diagram the stress at the breaking point appears much less than the maximum stress carried, but on examination of the actual stress diagram it is found to be the greatest stress.

32. **Important Stages in the Test.** All the important stages have been marked on the nominal stress-strain curve shown in Fig. 17, and a particular name is usually applied to the stress at each stage, as follows—

At the point *A* the elasticity has broken down, and the stress corresponding to the load at *A* is called the "Elastic Limit."

At the point *B* the proportionality of stress and strain has ceased, and the stress at *B* is called the "Limit of Proportionality."

At the point *C* the material has yielded a large amount, and the corresponding stress is known as the "Yield Stress."

These three stresses are exceedingly close together, and great care and delicate measurements are required to distinguish one from the other.

At the point D local yielding takes place, and the waist begins to form. The material is then carrying the maximum load, and the corresponding stress is known as the " Maximum Stress." It should be remembered, however, that this is maximum nominal stress.

At the point E fracture takes place, and the stress at this point is known as the " Breaking Stress."

33. **Working Stresses.** The "Tenacity" or "Ultimate Stress" of a material is the greatest load required to fracture the material divided by the area of the original cross-section at the point of fracture. It is important that the working stress shall be well below the elastic limit, and the ultimate stress is usually divided by a number called the *Factor of Safety* in order to obtain the working stress. The factor of safety has also been referred to as a factor of ignorance. It would seem more reasonable to use a smaller factor of safety and the elastic limit when calculating the working stress.

Factors of safety depend on how the stress is applied. An alternating stress requires a larger factor of safety than a constant stress, and a stress caused by shock will require the factor of safety to be much larger still. Since the various working stresses are calculated from factors of safety based on the ultimate stress, it can be readily understood that the ultimate stress is the most important stress in a commercial test, the elastic limit being only asked for on rare occasions.

34. **The Measurement of Ductility.** The elongation on a test bar, up to the maximum load, is practically distributed uniformly over the bar, after which local yielding occurs and the cross-section decreases at the point of yielding. Two methods are used to estimate the ductility, one being based on the total elongation, and the other on the total reduction in cross-sectional area. The former is the more common and from it we obtain the " Percentage Elongation." If L is the length of the test bar at fracture, and l the length before stress is applied, i.e. the original length, then

$$\text{Percentage elongation} = \frac{L - l}{l} \times 100$$

The percentage elongation depends on the length of the bar

owing to the local yielding before fracture. If the test bar be marked off in inches, then the percentage elongation of the inch containing the fracture will be very large. If, however, the result is calculated on a 2-inch length containing the fracture, the local extension does not play so important a part, and the importance of its effect becomes less as the length of the test bar increases. It is of the utmost importance, therefore, always to state the length of the test bar on which the percentage elongation has been calculated. A common length of test bar is 8 in. The following table serves to show how the percentage elongation varies with the length of the specimen. The results are taken from a test on a mild steel bar.

Original length (inches)	0·5	1·5	2·5	3·5	4·5	5·5	6·5	8
Final length (inches)	0·9	2·2	3·48	4·65	5·88	7·08	8·28	10·1
Percentage elongation	80	46·7	39·2	32·8	30·6	28·7	27·4	26·2

The percentage elongation is usually calculated without taking into account the cross-sectional area of the test bar, although the results are not strictly comparable when the cross-sectional dimensions differ for a given gauge length. Where bars have different cross-sectional areas, Prof. Unwin* has shown that comparative results may be obtained if the bars are geometrically similar.

The total extension of a bar is made up of a uniformly distributed extension and a local extension, the former being proportional to the length of the bar, and the latter almost independent of the length.

Thus if x is the total extension, y the local extension, and l the gauge length, then—

$$x = y + zl.$$

Now y is nearly proportional to the square root of the cross-sectional area of the bar. Hence if A is the cross-sectional area

$$y = s\sqrt{A}$$

$$\therefore\ x = s\sqrt{A} + zl.$$

* *Proc. Inst. C.E.*, Vol. clv.

Thus, if s and z are known, a rough idea can be obtained of the elongation of another bar of different dimensions and the same material.

EXAMPLE 1.

A bar of wrought iron 0·875 in. diam. and 8 in. gauge length was subjected to a static tensile test. The results gave yield load 8·7 tons, maximum load 13 tons, and load at instant of fracture 11 tons. The total elongation was 2·3 in., of which 1·8 in. was elongation up to the point of maximum load. The area of reduced section at fracture was 0·315 sq. in. From the above calculate the results of a tensile test on a bar of the same material 1 in. diam. and 6 in. gauge length, giving the elongation per cent, and the load at instant of fracture. (Lond. Univ., 1919.)

Total extension $x = s\sqrt{A} + zl$

$$\text{and } s\sqrt{A} = 2{\cdot}3 - 1{\cdot}8 = 0{\cdot}5 \text{ in.}$$

$$\therefore s = \frac{0{\cdot}5}{\sqrt{0{\cdot}6013}} = \frac{0{\cdot}5}{0{\cdot}7754} = 0{\cdot}6449$$

$$\text{also } zl = 1{\cdot}8 \text{ in.}$$

$$\therefore z = \frac{1{\cdot}8}{8} = 0{\cdot}225$$

Total extension on 6 in. gauge length and 1 in. diameter specimen

$$= 0{\cdot}6449 \sqrt{0{\cdot}7854} + 0{\cdot}225 \times 6$$

$$= 0{\cdot}5715 + 1{\cdot}35$$

$$= 1{\cdot}9215 \text{ in.}$$

$$\text{Percentage elongation} = \frac{L - l}{l} 100 = \frac{1{\cdot}9215}{6} \times 100$$

$$= 32{\cdot}025 \text{ per cent}$$

yield load on 1 in. specimen = yield stress × area of section

$$= \frac{8{\cdot}7}{0{\cdot}6013} \times 0{\cdot}7854$$

$$= 11{\cdot}36 \text{ tons}$$

Up to the maximum load the elongation is uniform along the bar, and the area of the cross-section of bar at this load is found from

Length × area of section = original length × original area of section

∴ Area of section of original bar at max. load

$$= \frac{8 \times 0{\cdot}6013}{9{\cdot}8} = 0{\cdot}491 \text{ sq. in.}$$

Area of section of 1 in. dia. bar at max. load

$$= \frac{6 \times 0{\cdot}7854}{7{\cdot}35} = 0{\cdot}641 \text{ sq. in.}$$

Maximum load on 1 in. dia. specimen = max. stress × area of section

$$= \frac{13}{0{\cdot}491} \times 0{\cdot}641$$

$$= 16{\cdot}97 \text{ tons}$$

Percentage reduction of area of original test bar

$$= \frac{0{\cdot}6013 - 0{\cdot}315}{0{\cdot}6013} \times 100 = 47{\cdot}62 \text{ per cent}$$

Reduction in area of 1 in. dia. specimen

$$= 0{\cdot}4762 \times 0{\cdot}7854$$

$$= 0{\cdot}3740 \text{ sq. in.}$$

∴ Area at fracture $= 0{\cdot}7854 - 0{\cdot}3740$

$$= 0{\cdot}4114 \text{ sq. in.}$$

Load at instant of fracture on 1 in. dia. specimen

= breaking stress × area of section

$$= \frac{11}{0{\cdot}315} \times 0{\cdot}4114$$

$$= 14{\cdot}35 \text{ tons}$$

The second method of calculating the ductility of a metal depends on the change in the cross-sectional dimensions, and from it is obtained the "Percentage Reduction in Area."

If A = area of cross-section before applying load

A_1 = ,, ,, ,, after fracture,

then percentage reduction in area $= \frac{A - A_1}{A} \times 100.$

35. Resilience. The term "resilience" is usually restricted to mean the amount of energy stored up in an elastic strained body per unit volume. In the case of a body strained, within the elastic range, in simple tension we have already seen that the load is proportional to the extension. The work done on the bar, which is equal to the strain energy, is given by

$$\frac{\text{load} \times \text{extension}}{2} = \frac{AB \times OB}{2} \text{ (Fig. 18).}$$

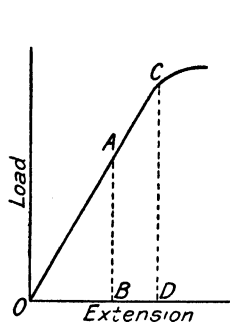
The term "proof resilience" is often adopted to represent the greatest strain energy per unit volume which can be stored, inside the elastic range, and is represented by the area

$$\frac{OCD}{\text{volume of bar}} = \frac{CD \times OD}{2 \text{ volume}}$$

Fig. 18

Let A = cross-sectional area of bar

l = length of bar

V = volume of bar

W = load applied

x = extension due to W

E = Young's modulus

f = tensile stress in bar due to W

Work done in straining bar

$$= \frac{W \times x}{2}$$

$$= \frac{fA \times x}{2}$$

$$= \frac{fA}{2} \frac{fl}{E}$$

$$= \frac{f^2}{2E} V$$

$\therefore$ Resilience $= \frac{f^2}{2E}$ (units of work per unit volume of bar).

If f_{max} is the greatest stress that can be applied within the elastic range, then

$$\text{Proof resilience} = \frac{1}{2}\frac{(f_{max})^2}{E}$$

The expression $\frac{(f_{max})^2}{E}$ is sometimes called the "Modulus of Resilience."

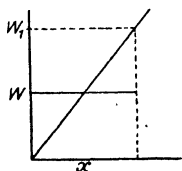
36. **Total Work done up to Fracture.** The total work done up to fracture is given by the area $OCDEF$ (Fig. 19). The portion OCH represents the work done during the elastic stage, and the remaining portion that done during the plastic stage.

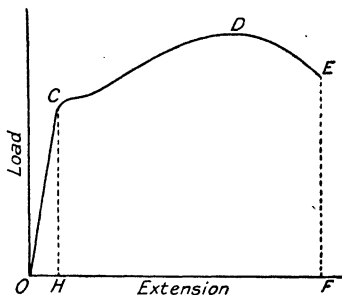
Fig. 19

37. **Suddenly-applied Loads.** Let a tensile load W be applied suddenly to a bar and produce an extension x. The work done will be given by Wx. Suppose W_1 to be a load which increases gradually from zero and produces the same extension, the work done is now given by $\frac{1}{2}W_1x$ (Fig. 20).

Since the bar is strained an equal amount in each case, the strain energy is the same in each case, and consequently the applied energies will be equal.

Fig. 20

$$\therefore \tfrac{1}{2}W_1x = Wx$$
$$W_1 = 2W$$

That is, the suddenly-applied load required to produce a given strain is just half the magnitude of that required, for the same strain, when gradually applied. This may also be stated in the form that the strain, and consequently the stress, produced by a given suddenly-applied load is exactly twice that produced by the same load when gradually applied. This result makes it clear that great danger arises from the application of a suddenly applied load to a machine part unless the part has been specially designed to withstand this type of load.

If a dead load of magnitude W_1 be carried by a bar of

cross-section A and a suddenly-applied load of magnitude W be applied, then the instantaneous stress is given by $\frac{W_1 \pm 2W_2}{A}$, *the upper sign being taken when the loads act in the same direction, and the lower sign when they act in opposite* directions.

38. **Stress Due to Shock.** Fig. 21 represents a weight W, which can fall freely on to a collar or anvil carried at the lower extremity of a vertical rod, the upper end of the rod being fixed. If the weight is allowed to fall through a height h and impinge on the collar, we can calculate the maximum stress in the rod when the following assumptions are made—

FIG. 21

(1) That the weight W falls freely.

(2) That the support is rigid.

(3) That there is no loss of energy by straining of the striking surfaces. (This assumption is not true in practice.)

The energy possessed by W at the height h is therefore expended in straining the rod.

Let x = the elongation of the rod

f = the maximum stress induced in the rod

A = cross-sectional area of rod

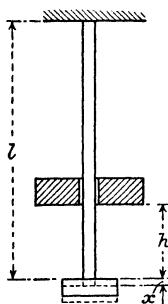
$$W(h + x) = \tfrac{1}{2}fAx$$

$$W\left(h + \frac{fl}{E}\right) = \tfrac{1}{2}f^2\frac{Al}{E}$$

$$\text{or } f^2\frac{Al}{2E} - f\frac{Wl}{E} - Wh = 0$$

$$\therefore f = \frac{\frac{Wl}{E} + \sqrt{\frac{W^2l^2}{E^2} + 2\frac{WhAl}{E}}}{\frac{Al}{E}}$$

$$\text{or } f = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2hAE}{Wl}}\right)$$

If $h = 0$, that is if the load is suddenly applied, then $f = \frac{2W}{A}$, a result already obtained in the previous paragraph.

EXAMPLE 2.

Define "Resilience." A weight of 20 lb. falls freely through 10 ft. and is then suddenly checked by the reaction of a bar of steel $\frac{3}{4}$ in. diameter, and 30 ft. long. Find the maximum stress and strain induced in the bar. $E = 30{,}000{,}000$ lb./sq. in. (Lond. Univ., 1912.)

$$f = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2hAE}{Wl}}\right)$$

$$= \frac{20}{0{\cdot}4418}\left(1 + \sqrt{1 + \frac{2 \times 120 \times 0{\cdot}4418 \times 30 \times 10^6}{20 \times 360}}\right)$$

$$= 45{\cdot}2 \times 665{\cdot}6$$

$$= 30{,}080 \text{ lb./sq. in.}$$

$$W(h + x) = \tfrac{1}{2}fAx$$

$$x = \frac{Wh}{\frac{1}{2}fA - W}$$

$$= \frac{20 \times 120}{\frac{1}{2} \times 30{,}080 \times 0{\cdot}4418 - 20}$$

$$= \frac{2400}{6646 - 20} = \frac{2400}{6626}$$

$$= 0{\cdot}3621 \text{ in.}$$

$$\text{Strain} = \frac{0{\cdot}3621}{360}$$

$$= 0{\cdot}001005$$

EXAMPLES III

1 Calculate the extension produced and the work expended in extension when a bar of steel 15 in. long and $1\frac{1}{2}$ in. diameter is subjected to a pull of 6 tons. Over what length must the bar be reduced to 1 in. diameter in order that the extension produced on the 15-in. length by the 6-ton load shall be increased by 50 per cent? $E = 30{,}000{,}000$ lb. per sq. in. (Lond. Univ., 1923.)
Ans., 0·003817 in., 25·65 in. lbs./6·05 in.

2. (*a*) How are the results of an ordinary tensile test of a sample of mild steel affected, as regards ultimate strength and percentage contraction of area of section, by the proportions of the test piece; you may confine your remarks to cylindrical test pieces.

(*b*) Explain how the percentage of ultimate elongation varies with the gauge length adopted and the diameter of a cylindrical test piece. What precautions would you adopt if you were obliged to compare the percentage ultimate extension of a sample of mild steel of which only a rather small cylindrical specimen was available, with a result obtained from a larger cylindrical test piece of a previous sample of material. (A.M.I.Mech.E., 1925.)

3. Tension experiments upon a set of wrought-iron bars showed that permanent set began when the stress exceeded 20,000 lb. per sq. in. The average extension per 10-in. length under the above stress of 20,000 lb. was 0·006 in. Determine the modulus of resilience in ft. lb. per cub. in. for this quality of wrought iron. (Lond. Univ., 1915.) *Ans.*, 1 ft. lb./cub. in.

4. A mild steel bar 1 in. diameter was subjected to a tensile test to destruction. The results were: yield load 17·2 tons, maximum load 22·7 tons, load at instant of fracture 19·5 tons. Extension measured on a 6-in. original length 2·05 in., and on 8 in. original length 2·5 in.

Find the work per cub. in. up to maximum load (i.e. just before the local contraction of area begins). Find also the extension you would expect from a test piece $1\frac{1}{2}$ in. diameter of the same material measured on a gauge length of 8 in. (Lond. Univ., 1921.) *Ans.*, 5·7 in. tons/2·855 in.

5. Differentiate clearly between the "limit of elasticity" of a metal and its "limit of proportionality." State what is meant by the yield point of a piece of mild steel. State what arguments may be adduced against fixing the allowable stress on a piece of steel as a proportion of the stress at the limit of elasticity. (A.M.I.Mech.E., 1926.)

6. What do you understand by the term "Modulus of Resilience" of a material? A bar 12 in. long is 1·25 in. in diameter for 5 in. of its length, and 1 in. in diameter for the remainder. The bar receives a blow in an axial direction, which induces in it a maximum stress of 15 tons per sq. in. Calculate the stress that would be induced in a bar of the same material 12 in. long and 1 in. in diameter throughout, if subjected to the same blow. (A.M.I.Mech.E., 1924.) *Ans.*, 13·82 tons/sq. in.

7. Two round bars A and B, of the same material, are each 10 in. long. A is 1 sq. in. in section for 4 in. of its length, and the remainder is 1·5 sq. in. in section. B is 1 sq. in. in section from end to end. If B receives an axial blow which induces a stress of 5 tons per sq. in., find the maximum stress produced by the same blow on A. Compare the "proof resilience" of the two bars. (A.M.I.Mech.E., 1924.) *Ans.*, 5·58 tons/sq. in. $\frac{A}{B} = \frac{8}{13}$.

8. Explain how a load, suddenly applied, but without impact, will produce an intensity of stress twice as great as the intensity of stress produced by the same load applied gradually. A bar, 20 ft. long, is suspended vertically, and a collar is fixed to the lower end. A weight of 200 lb. is threaded on the bar, and falls a distance of 6 in. before engaging with the collar. Determine the diameter of the bar if the intensity of stress induced is not to exceed 12,000 lb. per sq. in. $E = 30{,}000{,}000$ lb. per sq. in. (A.M.I.Mech.E., 1918.)
Ans., 1·64 in.

9. Find the cross-sectional area of a vertically suspended steel tie-bar 6 ft. long, subjected to a load of 0·5 ton, which is allowed to fall 0·25 in. on to a collar at the bottom end of the bar. The ratio of the extension to the original length must not exceed $\frac{1}{1800}$. $E = 12{,}000$ tons per sq. in. (A.M.I.Mech.E., 1919.) *Ans.*, 1·09 sq. in.

10. A sliding weight of 4,000 lb. is dropped down a vertical rod which is suspended from the top, and is provided with a collar at the bottom end. The length of the rod is 12 ft. and the diameter is 2 in. In order to reduce the shock a helical buffer spring is placed on the collar; the spring will compress 1 in. per 1,000 lb. dead load. Taking account of the work done in compressing

the spring and in stretching the bar, find approximately the height, measured from the top of the uncompressed spring, from which the weight must be dropped in order to produce a momentary stress of 10,000 lb. per sq. in. in the bar. Young's modulus = 30,000,000 lb. per sq. in. (Lond. Univ., 1913).

Ans., 92 in.

11. A bar $\frac{3}{4}$ in. diameter and 8 ft. long is rigidly fixed at one end and is used to stop suddenly a weight W falling in the direction of the length of the bar. If W falls 2 ft. before coming to rest, find its magnitude if the stress in the bar due to the blow is not to be more than 7 tons per sq. in. $E = 13{,}000$ tons per sq. in. (Lond. Univ., 1916.) *Ans.*, 7·40 lb.

12. A crane chain whose sectional area is 1 sq. in. carries a load of 2,000 lb., which is being lowered at a uniform rate of 120 ft. per min. When the length of the chain unwound is 30 ft. the chain suddenly jambs on the pulley. Estimate the stress induced in the chain (neglecting the weight of the chain) due to the sudden stoppage. Modulus of direct elasticity = 30×10^6 lb./sq. in. $g = 32{\cdot}2$ ft./sec./sec. (A.M.I.C.E., 1925.) *Ans.*, 15,800 lb./sq. in.

CHAPTER IV

THIN CYLINDERS, SPHERES, AND PIPES

39. **Thin Cylinders.** Vessels such as steam boilers and large pipes are subjected to an internal fluid pressure which is uniformly distributed over the internal surfaces. Such vessels have large cross-sectional dimensions in comparison with the thickness of their plates. In general there is a principal stress acting in the direction tangential to the circumference, called the "hoop stress," a principal stress acting in the direction of the axis, and a principal stress acting in a radial direction. The last-mentioned stress can be neglected, however, where the thickness of the plates is small compared to the cross-sectional dimensions, also the first principal stress which is not uniform, over a section from the internal surface to the external surface, may be taken as uniform under this condition. These assumptions lead to the following easy method of attack—

(*a*) *Hoop Stress.* In Fig. 22A let r be the internal radius of the cylinder, subjected to a uniform internal fluid pressure of intensity p. Consider the equilibrium of a length of the cylinder between the parallel sections AA and BB. The radial force acting on the small area subtended by the angle $\delta\theta$ $= plr\delta\theta$, and the component of this force perpendicular to a diameter is $plr \sin \theta\delta\theta$.

The total force perpendicular to a diameter

$$= \int_0^\pi plr \sin \theta d\theta$$

$$= 2plr.$$

If f_v is the intensity of the "hoop stress," then the

$$\text{Resisting force} = 2F_v$$

$$= 2f_v l \times t$$

Hence for equilibrium of the material,

$$2f_v l \times t = 2plr$$

$$\therefore f_v = \frac{pr}{t} \quad . \quad . \quad . \quad (1)$$

(*b*) ***Stress in the Direction of the Axis.*** If the ends of the cylinder are only constrained by the material, there will be a stress f_x in the shell acting in the direction of the axis, as shown in Fig. 22B. The total force on the ends, whether plane or dished, acting axially due to the internal fluid pressure will be $\pi r^2 p$, and the resisting force $= 2\pi r t f_x$. Therefore for equilibrium of the material

$$2\pi r t f_x = \pi r^2 p$$

$$\therefore f_x = \frac{pr}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

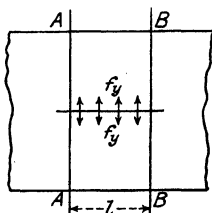



Fig. 22A

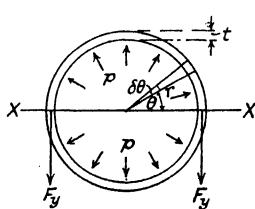



Fig. 22B

Thus the intensity of stress, in the shell, acting in the direction of the axis is only half the intensity of the hoop stress, and these are the principal stresses in the material.

EXAMPLE 1.

A steam boiler, 10 ft. internal diameter, with flat ends (not stayed), plates $\frac{3}{4}$ in. thick, sustains an internal pressure of 200 lb./sq. in. Calculate (1) the stress in the circumferential plates due to the load on the end plates, (2) the stress in the circumferential plates due to resisting the bursting effect, (3) the maximum shear stress due to the combined action of (1) and (2). (Lond. Univ., 1919.)

$$f_x = \frac{pr}{2t} \qquad (1)$$

$$= \frac{200 \times 60 \times 4}{2 \times 3}$$

$$= 8000 \text{ lb./sq. in.}$$

$$f_y = \frac{pr}{t} \qquad (2)$$

$$= 2f_x$$

$$= 16{,}000 \text{ lb./sq. in.}$$

(3) The maximum shear stress is given by

$$q = \frac{f_y - f_x}{2}$$

$$= \frac{16{,}000 - 8000}{2}$$

$$= 4000 \text{ lb./sq. in.}$$

40. **Modification for Built-up Shell.** The foregoing argument assumes that the shell and end plates are of jointless uniform material. Steam boilers and large pipes are, however, built up of plates joined by riveted joints, and hence the preceding equations require slight modification for such cases.

The resisting force of the boiler shell will be reduced by an amount depending on the efficiency of the joint, and the new resisting force is now equal to the previous value multiplied by e where e is the joint efficiency. The equations (1) and (2) now give

$$f_y = \frac{pr}{te} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{and } f_x = \frac{pr}{2te} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

41. **Application of Compound Stress Theories.** (*a*) If f is the yield stress for the material, then, according to the maximum principal stress theory,

$$f = f_y$$

and thus the greatest allowable pressure is given by

$$p = \frac{t}{r}f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(*b*) According to the maximum principal strain theory, the stress corresponding to the maximum principal strain is given by

$$Ee_{d_y} = f_y - \frac{f_x}{m} = f_y\left(1 - \frac{1}{2}\cdot\frac{1}{m}\right), \text{ par. (23)}$$

$$= \tfrac{7}{8} f_y \text{ for a value of } m = 4$$

$$\therefore f = \tfrac{7}{8} f_y$$

$$= \frac{7}{8}\frac{pr}{t}$$

$$\therefore p = \frac{8}{7}\frac{t}{r}f \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It would appear from this theory, therefore, that the shell is strengthened in a circumferential direction by the application of the stress acting in the direction of the axis.

(c) From Haigh's strain energy theory we have

$$\frac{f^2}{2E} = \frac{1}{2E}\left\{f_x^2 + f_y^2 - \frac{2f_x f_y}{m}\right\} \text{ by par. 28}$$

$$\therefore f^2 = \tfrac{1}{4}f_y^2 + f_y^2 - \frac{f_y^2}{m}$$

$$= f_y^2\left(\frac{1}{4} + 1 - \frac{1}{m}\right)$$

$$= f_y^2 \text{ for } m = 4.$$

$$\therefore f = f_y$$

$$\text{and } p = \frac{t}{r}f \qquad \qquad (7)$$

Thus this theory gives, for a value of $m = 4$, a value of p, similar to that given by the maximum principal stress theory.

EXAMPLE 2.

A marine boiler is 15 ft. in diameter, and is required to carry a working pressure of 180 lb./sq. in. The specified tensile stress in the plates is not to exceed 5 tons/sq. in. and the efficiency of the riveted joints is to be taken at 80 per cent. What thickness of plates would be necessary ? (A.M.I.Mech.E., 1918.)

The boiler plates must be designed to carry the maximum stress $f_y = \dfrac{pr}{t}$

and taking the joint efficiency into account, this becomes

$$f_y = \frac{pr}{te}$$

$$\therefore t = \frac{pr}{f_y \times e}$$

$$= \frac{180 \times 15 \times 12 \times 100}{2 \times 5 \times 2240 \times 80}$$

$$= 1\tfrac{7}{8} \text{ in.}$$

EXAMPLE 3.

A long straight tube, 3 in. internal diameter and $\frac{1}{10}$ in. thick, is subjected to an internal pressure of 800 lb./sq. in. Consider it as a thin cylinder and find the longitudinal and circumferential stresses. If the tube is then subjected to a twisting moment of 700 in. lb., find the factors of safety on the three recognized theories of elastic breakdown. Assume that Poisson's ratio = 0·3; elastic limit stress = 18 tons per sq. in. $E = 30 \times 10^6$ lb per sq. in. (Lond. Univ., 1918.)

Stress acting in a longitudinal direction

$$= \frac{pr}{2t} = \frac{800 \times 3 \times 10}{2 \times 2}$$

$$= 6{,}000 \text{ lb./sq. in.}$$

Stress acting in a circumferential direction

$$= \frac{pr}{t} = \frac{800 \times 3 \times 10}{2}$$

$$= 12{,}000 \text{ lb./sq. in.}$$

Shear stress due to torsion

$$= \frac{2TR}{\pi\,(R^4 - r^4)} \text{ see par. 108 (1)}$$

$$= \frac{2 \times 700 \times 3{\cdot}2 \times 8}{\pi\,(\,(3{\cdot}2)^4 - 3^4)}$$

$$= 476 \text{ lb./sq. in.}$$

Maximum principal stress

$$= \tfrac{1}{2}\{(f_x + f_y) + \sqrt{(f_x - f_y)^2 + 4f_s^2}\}$$

$$= \tfrac{1}{2}\{(12{,}000 + 6000) + \sqrt{(6000)^2 + 4(476)^2}\}$$

$$= \tfrac{1}{2}\{18{,}000 + 6075\}$$

$$= 12{,}038 \text{ lb./sq. in.}$$

$\therefore$ Factor of safety on maximum principal stress theory

$$= \frac{18 \times 2240}{12{,}038}$$

$$= \mathbf{3{\cdot}35}$$

The maximum principal strain

$$= \frac{1}{2E}\left\{(f_x + f_y)\left(1 - \frac{1}{m}\right) + \left(1 + \frac{1}{m}\right)\sqrt{(f_x - f_y)^2 + 4f_s^2}\right\}$$

$$= \frac{1}{2E}\left\{(18{,}000 \times 0\cdot7) + (1\cdot3 \times 6075)\right\}$$

$$= \frac{1}{2E} \times 20{,}500 = \frac{10{,}250}{E}$$

In simple tension the maximum strain $= \dfrac{18 \times 2240}{E}$

∴ Factor of safety on the maximum strain theory

$$= \frac{18 \times 2240}{E} \times \frac{E}{10{,}250} = 3\cdot93$$

The minimum principal stress is

$$= \tfrac{1}{2}\ \{18{,}000 - 6075\} = 5963 \text{ lb./sq. in.}$$

The principal stresses are thus, 12,038 lb./sq. in., 5,963 lb./sq. in. and 800 lb./sq. in. Hence, by par. 16 (12), the maximum shear stress in the material

$$= \frac{12{,}838}{2} = 6419 \text{ lb./sq. in.}$$

The maximum shear stress due to a simple tension of 18 tons/sq. in.

$$= \frac{18 \times 2240}{2}$$

∴ Factor of safety according to the maximum shear theory

$$= \frac{18 \times 2240}{2 \times 6419} = 3\cdot14$$

The factor of safety according to the strain energy theory may be found as follows—

$$f^2 = f_1^2 + f_2^2 - \frac{2f_1f_2}{m} = 12038^2 + 5963^2 - 2 \times 12038 \times 5963 \times 0\cdot3$$

$$\therefore f = 11{,}722 \text{ lb./sq. in.}$$

Factor of safety $= \dfrac{18 \times 2240}{11{,}722} = 3\cdot44$

42. **Increase in Volume of Cylinder.** The strain in the direction of the axis is given by

$$e_{d_x} = \frac{f_x}{E} - \frac{f_y}{mE}$$

$$= \frac{f_x}{E}\left(1 - \frac{2}{m}\right)$$

$$= \frac{pr}{2tE}\left(1 - \frac{2}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the strain in the direction of the hoop stress, or circumferential strain, by

$$e_{d_y} = \frac{f_y}{E}\left(1 - \frac{1}{2m}\right)$$

$$= \frac{pr}{tE}\left(1 - \frac{1}{2m}\right) \quad . \quad . \quad . \quad (9)$$

Now the increase in radius $= r \times$ circumferential strain. Hence the radius increases in the same proportion as the circumference.

By par. 7 the volume strain is given by

$$e_v = e_x + e_y + e_z$$

$$= e_{d_x} + 2\, e_{d_y}$$

$$= \frac{pr}{tE}\left\{\frac{1}{2} - \frac{1}{m} + 2 - \frac{1}{m}\right\}$$

$$= \frac{pr}{tE}\left\{\frac{5}{2} - \frac{2}{m}\right\}$$

and the increase in volume is therefore

$$\delta V = V\frac{pr}{tE}\left(\frac{5}{2} - \frac{2}{m}\right) \quad . \quad . \quad . \quad . \quad (10)$$

where V is the original volume of the cylinder.

43. **Thin Spherical Shell.**

Let r = radius of shell

t = uniform thickness of the material

p = internal pressure in shell.

The shell will tend to fail along a diametral plane such as EE, Fig. 23.

The force acting on a thin ring of internal surface $ABCD$ $= 2\pi r \cos \theta\, r\, \delta\theta p$.

The vertical component of this force

$$= 2\pi pr^2 \sin \theta \cos \theta\delta\theta$$

$$= \pi r^2 p \sin 2\theta\delta\theta.$$

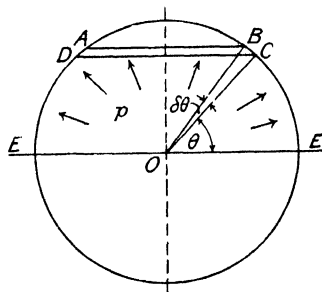



Fig. 23

The total force tending to cause failure along EE

$$= \pi r^2 p \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta$$

$$= \pi r^2 p.$$

This force is resisted by a force $= 2\pi rtf_z$ where f_z is the stress in the material.

$$\therefore 2\pi rtf_z = \pi r^2 p$$

$$\text{and } f_z = \frac{pr}{2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus the stress in a thin spherical shell is equal to the stress in the direction of the axis of a cylindrical shell. The principal stresses at any point in the shell are $\frac{pr}{2t}$, $\frac{pr}{2t}$, and a radial principal stress which may be neglected.

44. Modification for Built-up Shell. When the spherical shell is built of plates joined by riveted joints the resisting force is lessened in proportion to the joint efficiency just as in the case of the cylindrical shell.

A similar argument gives the modified formula as

$$f_z = \frac{pr}{2te} \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where e is the efficiency of the joint.

45. **Application of Compound Stress Theories.** (*a*) Let f = the yield stress of the material. According to the maximum principal stress theory,

$$f = f_z = \frac{pr}{2t}$$

and the maximum allowable pressure is given by

$$\therefore p = 2\frac{t}{r}f \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

(*b*) From the maximum principal strain theory the stress corresponding to the maximum principal strain is

$$Ee_z = f_z - \frac{f_z}{m}$$

$$= f_z\left(1 - \frac{1}{m}\right)$$

$$= \tfrac{3}{4} f_z \text{ for a value of } m = 4$$

$$\therefore f = \frac{3}{4} f_z = \frac{3}{4}\frac{pr}{2t}$$

$$\text{or } p = \frac{8}{3} \cdot \frac{t}{r} f \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Thus the maximum allowable pressure according to this theory is slightly larger than that given by the maximum principal stress theory.

(*c*) The strain energy theory gives—

$$\frac{f^2}{2E} = \frac{1}{2E}\left\{f_z^2 + f_z^2 - \frac{2f_z^2}{m}\right\}$$

$$\text{or } f^2 = f_z^2\left(2 - \frac{2}{m}\right)$$

$$= \frac{3}{2} f_z^2 \text{ when } m = 4$$

$$\therefore f = \sqrt{\frac{3}{2}} f_z = \sqrt{\frac{3}{2}} \frac{pr}{2t}$$

$$\text{and } p = 1{\cdot}635 \frac{t}{r} f \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This theory therefore proves the maximum allowable pressure to be smaller than that given by either of the previous theories.

46. Increase in Volume of Sphere. The circumferential strain is

$$e_{d_z} = \frac{f_z}{E} - \frac{f_z}{mE}$$

$$= \frac{pr}{2tE}\left(1 - \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$= \frac{3}{8} \frac{pr}{tE} \text{ when } m = 4,$$

and since the radius increases in the same proportion as the circumference,

The volumetric strain $= 3e_{d_z}$

$$= \frac{3}{2} \frac{pr}{tE}\left(1 - \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the increase in volume is therefore

$$\delta V = \frac{3}{2} \frac{pr}{tE}\left(1 - \frac{1}{m}\right) V \quad . \quad . \quad . \quad (8)$$

where V is the original volume of the sphere.

Example 4.

A spherical vessel 2 ft. diameter and $\frac{1}{4}$ in. thick is filled with a fluid until its volume increases by 2 cub. in. What is the pressure exerted by the fluid on the vessel? $m = 4$, $E = 30 \times 10^6$ lb./sq. in.

$$\delta V = \frac{3}{2} \frac{pr}{tE}\left(1 - \frac{1}{m}\right) V$$

$$2 = \frac{3}{2} \times \frac{p \times 12 \times 4 \times 3 \times 4\pi \times 1728}{30 \times 10^6 \times 4 \times 3}$$

$$p = 152 \text{ lb./sq. in.}$$

In the previous example, if the yield stress of the material is 18 tons/sq. in., what would be the maximum pressure before yield, according to the following theories : (*a*) Maximum stress, (*b*) maximum strain, (*c*) strain-energy ?

$$f_s = \frac{pr}{2t} = f$$

$$(a) \qquad \therefore p = 2\frac{t}{r}f = \frac{2 \times \frac{1}{4}}{12} \times 18 \times 2240$$

$$= 1680 \text{ lb./sq. in.}$$

$$(b) \qquad p = \frac{8}{3}\frac{t}{r}f = \frac{8}{3} \times \frac{\frac{1}{4}}{12} \times 18 \times 2240$$

$$= 2240 \text{ lb./sq. in.}$$

$$(c) \qquad p = 1{\cdot}635\frac{t}{r}f = 1{\cdot}635 \times \frac{\frac{1}{4}}{12} \times 18 \times 2240$$

$$= 1370 \text{ lb./sq. in.}$$

47. **Effect of Hemispherical Ends on Cylindrical Shell.** Serious bending stresses are introduced at the junction of the ends and shell, if the thickness of each is not properly proportioned, due to unequal radial expansion. Such a case occurs, as will be shown below, when the ends and shell are of equal thickness. It will also be shown, however, that if radial expansion is equal for each, then the ends will be weaker than the shell, and thus the working pressure is determined by the ends according to one of the theories of elastic failure.

Let p = internal fluid pressure

r = radius of hemispherical ends and shell

t_s = thickness of hemispherical ends

t_c = " " cylindrical shell.

In par. 46 (6) it was shown that the strain in the direction of the radius of a sphere was given by

$$\frac{pr}{t_sE}\left(\frac{m-1}{2m}\right) \text{ when free from restraint.}$$

Also the strain in a similar direction for a cylindrical shell when unrestrained is given in par. 42 (9) as

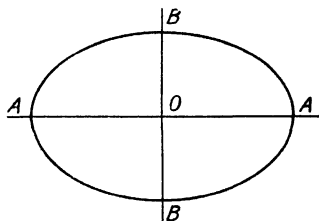
$$\frac{pr}{t_cE}\left(\frac{2m-1}{2m}\right)$$

Comparing these expressions, it will be clear that if t_s and t_c are equal, then the expansion of the shell will be greater than the expansion of the ends, and thus, if they are riveted together, serious bending stresses will be introduced.

To get the ratio between t_s and t_c, in order that the bending stresses may be avoided, we have

FIG. 24

$$\frac{pr}{t_cE}\left(\frac{2m-1}{2m}\right)=\frac{pr}{t_sE}\left(\frac{m-1}{2m}\right)$$

$$\therefore \frac{t_s}{t_c}=\frac{m-1}{2m-1}$$

which for $m = 4$ gives $t_s = \frac{3}{7}t_c$; this, however, unfortunately, makes the ends much weaker than the shell, and thus the allowable working pressure is reduced.

48. **Cylinders of Oval Section.** Thin cylinders of oval section, when subjected to internal fluid pressure, tend to assume a circular section. The intensity of hoop tension varies along the periphery. By a similar method to that employed in par. 39, the hoop stress at A, A (Fig. 24) can be shown to be $\frac{p \times OA}{t}$, and that at B, B, $\frac{p \times OB}{t}$, where p is the internal fluid pressure and t the uniform thickness of the material. Also if A is the area of the cross-section of the pipe, and a the area of the cross-section of the material of the pipe and f_x the stress in the direction of the axis of the pipe

$$f_x \times a = p \times A$$

$$f_x = p\frac{A}{a} \text{ or } = p\frac{A}{t \times l}$$

where l is the length of the perimeter $\doteqdot 1{\cdot}57\ (AA + BB)$.

49. **Thin Tubes Subjected to External Pressure.** The collapsing pressure of thin tubes under external pressure has been

investigated by Southwell and others, and the reader is referred to the *Phil. Mag.*, 1913, for a study of Southwell's investigation. He has derived the following formula for pipes whose thickness is small compared to the diameter—

$$p = 2E\frac{t}{d}\left\{\frac{S}{k^4\,(k^2-1)}\left(\frac{d}{l}\right)^4 + \frac{1}{3}\,\frac{(k^2-1)\,m^2}{m^2-1}\left(\frac{t}{d}\right)^2\right\}$$

where p = the collapsing pressure

t = thickness of the tube

l = the length of the tube

d = diameter of tube

S = constant depending on the end fastening

k = the number of lobes in the collapsed section.

When l is large in comparison to the diameter the first term is negligible and the formula reduces to

$$p = \frac{2Et}{d}\left\{\frac{1}{3}\,\frac{(k^2-1)\,m^2}{m^2-1}\left(\frac{t}{d}\right)^2\right\}$$

$$= \frac{2}{3}(k^2-1)\left(\frac{m^2}{m^2-1}\right)\frac{t^3}{d^3}E$$

Now the smallest number of lobes which can occur is 2, and thus

$$p = 2\,\frac{m^2}{m^2-1}\,\frac{t^3}{d^3}\,E, \text{ or if } m = 4$$

$$p = \frac{32}{15}\,\frac{t^3}{d^3}\,E.$$

EXAMPLES IV

1. What thickness of metal would be required for a cast-iron water pipe 36 in in diameter under a head of 400 ft. ? Assume a limiting tensile stress for cast iron of 3,000 lb./sq. in. (A.M.I.Mech.E., 1919.) *Ans.*. 1 in.

2. Show that, in the case of a thin cylindrical shell subjected to internal fluid pressure, the tendency to burst lengthwise is twice as great as at a transverse section. The shell of a boiler is 6 ft. in diameter, and the plates are 0·75 in. thick; the efficiency of the joints to tearing along a longitudinal seam being 72 per cent. Calculate the safe working pressure in the boiler, assuming that the tensile strength of the plates is 30 tons/sq. in. (take a factor of safety of 6). (A.M.I.Mech.E., 1924.) *Ans.*, 168 lb./sq. in.

3. A spherical vessel, built up of thin steel plates, has to withstand an internal pressure of 100 lb./sq. in. The internal diameter is 10 ft. and the joint efficiency is 80 per cent. Calculate the thickness of the plates required, and the internal diameter when the vessel is under pressure. Working stress 5 tons/sq. in., $m = 4$. *Ans.*, $\frac{3}{8}$ in., 120·024 in.

4. The following data refer to the shell of a boiler of the Scottish type: Internal diameter 16 ft., working steam pressure (above atmospheric) 210 lb./sq. in., thickness of shell plates $1\frac{13}{32}$ in., diameter of rivet holes $1\frac{3}{8}$ in. The longitudinal joint is a treble riveted double strap butt joint, the pitch of the rivets in the outer row being $10\frac{1}{2}$ in. The circumferential joint is a double lap joint, the pitch of the rivets being $4\frac{1}{2}$ in. Calculate the maximum stress in the boiler plate (*a*) longitudinally, and (*b*) circumferentially. (Lond. Univ., 1915.) *Ans.*, Hoop stress 15,000 lb./sq. in., in direction of axis 10,000 lb./sq. in.

5. A steel cylinder 6 in. internal diameter, with walls $\frac{1}{8}$ in. thick, is closed by rigid caps at its ends. The caps are held together by a steel stay of circular section 1 in. diameter passing through the caps. Pressure-tight joints are made by tightening nuts on the ends of the stays. The materials of the cylinder and the stay have substantially equal elastic constants and the effective length of the stay subjected to stress is 10 per cent greater than the length of the cylinder. Find by how much the force on each joint will be reduced if, after the joints are made tight, water at 200 lb./sq. in. pressure is introduced into the cylinder. Take Poisson's ratio as $\frac{1}{4}$. (A.M.I.Mech.E., 1925.) *Ans.*, 4,930 lb.

6. A cylinder is 10 ft. long, 3 ft. in diameter, and $\frac{1}{2}$ in. thick at atmospheric pressure. Calculate its dimensions when subjected to an internal pressure of 200 lb./sq. in. What is then the maximum shear stress in the shell? $E = 30 \times 10^6$ lb./sq. in., $m = 4$.

Ans., 120·0072 in., 36·00756 in., 3,600 lb./sq. in.

7. A thin cylindrical shell 8 ft. in diameter is composed of plates $\frac{1}{2}$ in. thick, these being joined by riveted joints whose efficiency is 84 per cent. The yield stress for the material is 20 tons/sq. in. What internal pressure would cause yielding (*a*) according to the maximum strain theory, (*b*) according to the strain energy theory?

Ans., (*a*) 448 lb./sq. in., (*b*) 392 lb./sq. in.

8. A steel pipe 12 in. diameter and $\frac{1}{4}$ in. thick is subjected to a pressure of 400 lb./sq. in. Estimate the hoop stress in the pipe under this pressure and also when wound, before stressing, with wire $\frac{1}{10}$ in. diameter carrying a tension of 10,000 lb./sq. in. What is the stress in the wire when the pipe is under pressure? *Ans.*, 9,600 lb./sq. in., 4,160 lb./sq. in., 17,300 lb./sq. in.

9. A thin spherical vessel 3 ft. in diameter and $\frac{1}{4}$ in. thick is filled with water. More water is pumped in until the pressure reaches 600 lb./sq. in. How much extra water was required to reach this pressure? $m = 4$, $E = 30 \times 10^6$ lb./sq. in. *Ans.*, 19·8 cub. in.

10. A cylindrical vessel composed of thin plates $\frac{3}{8}$ in. thick has a diameter of 4 ft., its ends being hemispherical. Estimate the thickness of the ends in order that the circumferential strain may be the same in the ends as the shell, and state the stress in the ends when the internal pressure is 150 lb./sq. in. $m = 4$. *Ans.*, $\frac{3}{16}$ in., 9,600 lb./sq. in.

11. A pressure gauge is formed of a copper tube of elliptical section whose major axis is $\frac{3}{4}$ in. and minor axis $\frac{1}{4}$ in. What will be the thickness of the material in order that the hoop stress may not exceed 5,000 lb./sq. in. under

an internal pressure of 1,000 lb./sq. in.? What is then approximately the stress in an axial direction? *Ans.*, $\frac{1}{20}$ in., 1,666·6 lb./sq. in.

12. A steel water pipe 24 in. diameter has to resist the pressure due to a head of 400 ft. of water; what thickness should it be made if the working intensity of pressure in the metal is to be 5 tons per square inch after the pipe has lost $\frac{1}{10}$ in. of its thickness, due to corrosion? (A.M.I.C.E., 1917.)
Ans., 0·285 in.

13. A steel tube is plugged at the ends and fitted with an extensometer to measure the axial elongations produced by—

(*a*) internal pressure,
(*b*) axial pull applied to the plugs.

When an axial pull of 3,000 lb. is applied to the plugs, the elongation of a 10 in. gauge length is 0·0033 in. The axial pull is removed and an internal pressure of 1,000 lb./sq. in. is applied. The elongation is then 0·00133 in.

The tube is 2 in. external diameter and 1·9 in. internal diameter. Deduce the value of Poisson's ratio for the material of the tube. (Lond. Univ., 1930.)
Ans., 0·3125.

14. Show that the strain of energy per unit volume of a material subjected to two principal stresses p and q is given by—

$$\frac{1}{2E}\left(p^2 + q^2 - \frac{2pq}{m}\right)$$

If a drum of 2 ft. internal diameter is to be subjected to an internal pressure of 200 lb./sq. in., calculate the thickness necessary to give a factor of safety of 2·5, if the criterion of failure is the maximum strain energy. The elastic limit in pure tension may be taken as 15 tons/sq. in. Poisson's ratio $\left(\frac{1}{m}\right) = 0{\cdot}3$. (Lond. Univ., 1937.) *Ans.*, 0·174 in.

15. A cylindrical steel vessel with hemispherical ends is 24 in. long over all, the outside diameter is 4 in. and the thickness 0·22 in. Calculate the change of internal volume of the vessel when it is subjected to an internal pressure of 2,000 lb./sq. in. Young's modulus $= 30 \times 10$ lb./sq. in. Poisson's ratio $=$ 0·288. (Lond. Univ., 1936.) *Ans.*, 0·218 cub. in.

CHAPTER V

CENTRE OF GRAVITY—MOMENT OF INERTIA AND ELLIPSE OF INERTIA

50. **Centroids.** The *Centre of Gravity or Centroid* of a body is that point in the body at which all its mass may be assumed to be concentrated ; that is, if the body were supported at the centroid it would remain in equilibrium. Similarly the centroid of a figure is that point in the figure at which all the area may be assumed to be concentrated.

If a small element of area a be at a perpendicular distance r from an axis, then ar is the first moment of the element a about the axis. If a figure consists of a number of small elements and the first moment of each of the elements be found, about a given axis, then the sum of the first moments of the elements is called the first moment of the area about the given axis and $= \Sigma ar$.

From the study of statics, we have the theorem that the moment of the resultant of a set of coplanar forces about a given point in the plane is equal to the sum of the moments of the forces about the same point. Suppose an area to be composed of small elements a_1, a_2, a_3, etc., then the resultant area $A = a_1 + a_2 + a_3$, etc. Let the figure have thickness t and the weight of unit volume be w.

Total weight $= A \times t \times w = wt\,(a_1 + a_2 + a_3 + \text{etc.})$.

Let h be the distance of the centroid from a given axis, and r_1, r_2, r_3, etc., the distance of the centroid of each small element from the same axis.

Then from the above theorem,

$$A \times t \times w \times h = wt\,(a_1r_1 + a_2r_2 + a_3r_3 + \text{etc.})$$

$$\therefore h = \frac{a_1r_1 + a_2r_2 + a_3r_3 + \text{etc.}}{A}$$

$$= \frac{\Sigma ar}{A}$$

Thus the distance of the centroid of a figure from a given axis is equal to the first moment of the area of the figure about the same axis, divided by the area of the figure. In order to

determine the centroid of a figure, the distance of the centroid, from two perpendicular lines on the figure, has usually to be found.

If the given axis, about which the first moment is taken, passes through the centroid, we have $\Sigma ar = Ah$, but h is now zero since the axis passes through the centroid, therefore $\Sigma ar = 0$, or the first moment of a figure about a line through its centroid is zero.

The figures dealt with in engineering practice can usually be split up into smaller figures such as rectangles, triangles, etc., whose centroids are known, and thus the calculations involved in finding the centroid of the original figure are usually of a simple nature. Where such is not the case the original figure can be divided up into thin strips, each of which approximates to a rectangle.

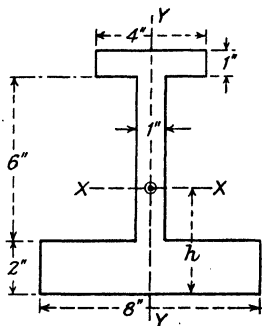



Fig. 25

Example 1.

The section of a cast-iron beam is shown in Fig. 25. Determine the centroid of the section.

Let XX be an axis parallel to the base and passing through the centroid o, and let YY be a perpendicular axis also passing through o. Since the figure is symmetrical about YY the centroid will be determined when the distance h of the line XX from the base is determined. Each flange and the web form rectangles, the centroids of which are at the intersection of the diagonals.

Area of upper flange $= 4 \times 1 = 4$ sq. in.
,, lower ,, $= 8 \times 2 = 16$,,
,, web ,, $= 6 \times 1 = 6$,,
Total area of section $= 26$ sq. in.

Distance of centroid of upper flange from base $= 8\frac{1}{2}$ in.
,, ,, ,, lower ,, ,, ,, $= 1$ in.
,, ,, ,, web from base $= 5$ in.

$$h = \frac{\Sigma ar}{A}$$

$$= \frac{(4 \times 8\frac{1}{2}) + (16 \times 1) + (6 \times 5)}{26} = \frac{80}{26}$$

$$= 3{\cdot}07 \text{ in.}$$

51. Moment of Inertia. If a small element, of area a, has its centroid at a distance r from a given axis, then the *Moment of Inertia*, or *Second Moment*, of the area about the given axis is the product of the element of area and the square of the distance of the centroid from the axis,

$$\text{or } I = ar^2$$

Let a figure of total area A, such as that shown by Fig. 26, be composed of small elements of area a such as that at S. Let OA and OB be two mutually perpendicular axes in the plane of the figure, and let I_1 and I_2 denote the moment of inertia of the figure about the axes OA and OB respectively, then

$$I_1 = \Sigma a \, . \, SC^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$I_2 = \Sigma a \, . \, SD^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

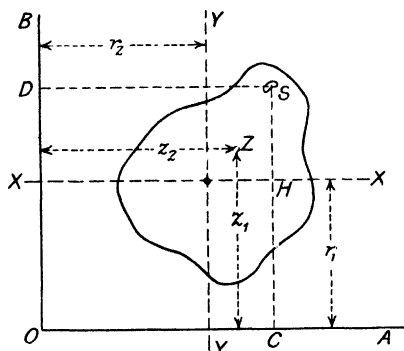



Fig. 26

Let Z be a point in the figure such that z_1 is the distance of Z from OA, and z_2 the distance of Z from OB, where z_1 and z_2 are obtained from

$$Az_1{}^2 = I_1 = \Sigma a \, . \, SC^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$Az_2{}^2 = I_2 = \Sigma a \, . \, SD^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then z_1 and z_2 are called the *radii of gyration* of the figure about the lines OA and OB.

$$\therefore z_1 = \sqrt{\frac{I_1}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$z_2 = \sqrt{\frac{I_2}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

52. **Moment of Inertia of Figure about a Parallel Axis through the Centroid.** Let XX and YY, Fig. 26, be lines through the centroid, and parallel to OA and OB respectively, also let I_{XX} and I_{YY} be the moment of inertia of the figure about XX and YY.

$$I_1 = \Sigma a \,.\, SC^2$$

$$= \Sigma a\,(SH + r_1)^2 = \Sigma a\,(SH^2 + 2r_1 SH + r_1^2)$$

$$= I_{XX} + 2r_1 \Sigma a\, SH + r_1^2 \Sigma a$$

$$= I_{XX} + 2r_1 \text{ (first moment of area about line through centroid)} + \text{total area } Ar_1^2$$

FIG. 27

or $$I_1 = I_{XX} + O + Ar_1^2$$

$$\therefore I_1 = I_{XX} + Ar_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and similarly

$$I_2 = I_{YY} + Ar_2^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the moment of inertia of a figure about an axis is equal to the moment of inertia of the figure about a parallel axis through the centroid plus the area of the figure multiplied by the square of the distance between the axes.

53. **Moment of Inertia of a Figure about Two Perpendicular Lines through the Centroid.** Let a figure be composed of small elements as a_1 (Fig. 27) and let d_1, b_1, and c_1, be the distances of the centroid of the element from the mutually perpendicular

axes GX, GY and GZ, which pass through the centroid of the figure, GX and GY being in the plane of the figure.

$$\text{Then} \quad I_{XX} = \Sigma a_1 d_1^2$$

$$I_{YY} = \Sigma a_1 b_1^2$$

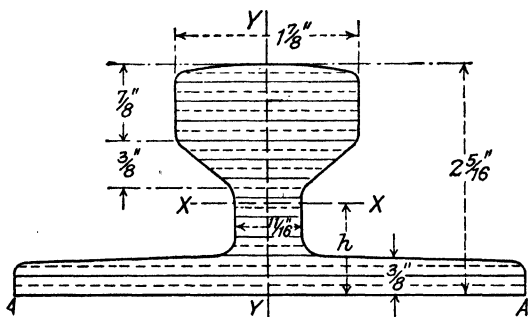



FIG. 28

$$\therefore I_{XX} + I_{YY} = \Sigma a_1 d_1^2 + \Sigma a_1 b_1^2$$

$$= \Sigma a_1 (d_1^2 + b_1^2)$$

$$= \Sigma a_1 c_1^2$$

$$= I_{ZZ}.$$

$$\therefore I_{ZZ} = I_{XX} + I_{YY} \quad . \quad . \quad . \quad . \quad (3)$$

or the sum of the moments of inertia of a figure about two perpendicular axes, through the centroid of the figure and in the plane of the figure, is equal to the moment of inertia of the figure about an axis perpendicular to the plane of the figure, and passing through the centroid. *It also follows from this that the sum of the moments of inertia of a figure about any two perpendicular axes through the centroid, and in the plane, of the figure, is constant.*

EXAMPLE 2.

Find the moment of inertia of the rail section, shown in Fig. 28, about the line XX passing through the centroid of the section and parallel to AA. Width of flange 5·25 ins.

The figure has been divided into parallel strips, each approximately 0·2 in. wide, by lines perpendicular to the axis of

symmetry YY. Take AA as an axis, then if x is the width of a strip, and y the distance of its centre line from AA, we have

Area of strip $= 0{\cdot}2x$.
First moment about $AA = 0{\cdot}2xy$.
Second moment about $AA = 0{\cdot}2xy^2$.
Area of section $= \Sigma 0{\cdot}2x$.
First moment of section about $AA = \Sigma 0{\cdot}2xy$.
$\therefore$ Distance of centroid of section from AA

$$= \frac{\Sigma 0{\cdot}2xy}{\Sigma 0{\cdot}2x}$$

Second moment of section about $AA = \Sigma 0{\cdot}2xy^2$.

The value of x and y have been measured for each strip, and the area a calculated, and from these the value of ay and ay^2 obtained. These are set out in tabular form below.

x = Width of Strip	a = Area of Strip	y = Distance of Centre of Strip from AA	ay = First Moment of Strip about AA	ay^2 = Second Moment of Strip about AA
1·85	0·370	2·3	0·8510	1·9573
1·88	0·376	2·1	0·7896	1·6582
1·88	0·376	1·9	0·7144	1·3574
1·88	0·376	1·7	0·6392	1·0866
1·60	0·320	1·5	0·4800	0·7200
1·15	0·230	1·3	0·2990	0·3887
0·70	0·140	1·1	0·1540	0·1694
0·69	0·138	0·9	0·1242	0·1118
0·69	0·138	0·7	0·0966	0·0676
0·73	0·146	0·5	0·0730	0·0365
5·18	1·036	0·3	0·3108	0·0932
5·25	1·050	0·1	0·1050	0·0105
Totals	4·696 $= \Sigma a$		4·6368 $= \Sigma ay$	7·6572 $= \Sigma ay^2$

Let h = distance of centroid of section from AA

$$h = \frac{\Sigma ay}{\Sigma a} = \frac{4{\cdot}6368}{4{\cdot}696} = 0{\cdot}9852 \text{ in.}$$

$$I_{AA} = \Sigma ay^2 = 7{\cdot}6572 \text{ inch units}$$

$$I_{XX} = I_{AA} - (\Sigma a)\, h^2 = 7{\cdot}6572 - 4{\cdot}6960 \times 0{\cdot}9852^2$$

$$= 7{\cdot}6572 - 4{\cdot}5580$$

$$= 3{\cdot}0992 \text{ inch units}$$

EXAMPLE 3.

Find the moment of inertia of each of the plane figures, shown in Fig. 29, about the axes XX and YY, which, in each case, pass through the centroid of the figure. Find also the moment of inertia about the line AA.

In par. 51, if the element of area a is made infinitely small and $SC = y$ and $SD = x$,

then $I_1 = \int y^2\,da$ and $I_2 = \int x^2\,da$.

(a) *Rectangle.*

$$I_{XX} = 2\int_0^{\frac{d}{2}} by^2\,dy = 2\left|\frac{by^3}{3}\right|_0^{\frac{d}{2}}$$

$$\text{or } I_{XX} = \frac{bd^3}{12} \text{ and similarly}$$

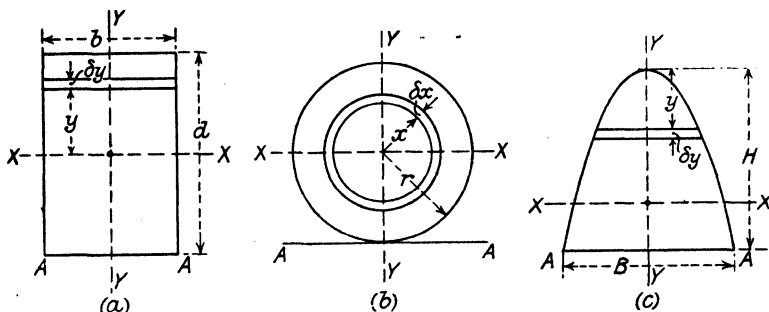



FIG. 29

$$I_{YY} = \frac{db^3}{12}$$

$$I_{AA} = I_{XX} + Ar^2 \text{ by (1), par. 52}$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4}$$

$$= \frac{bd^3}{4}\left(\frac{1}{3} + 1\right)$$

$$\therefore I_{AA} = \frac{bd^3}{3}$$

(*b*) *Circle.* This problem is perhaps best approached by finding first the moment of inertia about an axis ZZ through the centroid and perpendicular to the plane of the figure.

$$I_{ZZ} = \int_0^r 2\pi x \, x^2 dx = 2\pi \int_0^r x^3 dx$$

$$= 2\pi \left| \frac{x^4}{4} \right|_0^r$$

$$= \frac{\pi r^4}{2}$$

Now $I_{XX} = I_{YY}$ and $I_{ZZ} = I_{XX} + I_{YY}$.

$$\therefore 2I_{XX} = I_{ZZ} = \frac{\pi r^4}{2}$$

$$\therefore I_{XX} = \frac{\pi r^4}{4}$$

$$I_{AA} = I_{XX} + Ar^2$$

$$= \frac{\pi r^4}{4} + \pi r^4$$

$$\therefore I_{AA} = \frac{5\pi r^4}{4}$$

(*c*) *Parabola.* The law of the curve is given by $x^2 = 4ay$ for the given position, and its area is $\frac{2}{3}BH$.

$$I_{AA} = \int_0^H 2x \, (H - y)^2 \, dy$$

$$= 4\int_0^H a^{\frac12} y^{\frac12} \, (H^2 - 2Hy + y^2) \, dy$$

$$= 4a^{\frac12} \left| \frac{H^2 y^{\frac32}}{\frac32} - \frac{2HY^{\frac52}}{\frac52} + \frac{y^{\frac72}}{\frac72} \right|_0^H$$

$$= 8a^{\frac12} H^{\frac72} \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right)$$

$$= \frac{64a^{\frac12} H^{\frac72}}{105} \text{ and } B = 4a^{\frac12}H^{\frac12}$$

$$\therefore I_{AA} = \frac{16}{105} BH^3$$

The first moment of the area about AA is given by

$$Ah = \int_0^H 2x\,(H - y)\,dy$$

$$= 4\int_0^H a^{\frac{1}{2}}\,(Hy^{\frac{1}{2}} - y^{\frac{3}{2}})\,dy$$

$$= 4a^{\frac{1}{2}} \left| \frac{Hy^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}} \right|_0^H$$

$$= 8a^{\frac{1}{2}}\,H^{\frac{5}{2}} \left| \frac{1}{3} - \frac{1}{5} \right|$$

$$= \frac{4}{15}\,BH^2$$

$$\therefore h = \frac{\frac{4}{15}\,BH^2}{A} = \frac{\frac{4}{15}\,BH^2}{\frac{2}{3}\,BH}$$

$$= \frac{2}{5}\,H \text{ from } AA$$

$$\text{and } I_{XX} = I_{AA} - Ah^2$$

$$= \frac{16}{105}\,BH^3 - \frac{8}{75}\,BH^3$$

$$= \frac{8}{175}\,BH^3$$

EXAMPLE 4.

Find the moment of inertia of the section of a cast-iron beam, shown by Fig. 25, about XX and YY.

Moment of inertia of upper flange about the axis XX

$$= \frac{4 \times 1^3}{12} + \{4\,(5{\cdot}43)^2\}$$

$$= 0{\cdot}33 + 118$$

$$= 118{\cdot}33 \text{ inch units,}$$

Moment of inertia of bottom flange about the axis XX

$$= \frac{8 \times 2^3}{12} + \{16 \times 2{\cdot}07^2\}$$

$$= 5{\cdot}33 + 68{\cdot}5$$

$$= 73{\cdot}83 \text{ inch units.}$$

Moment of inertia of web about the axis XX

$$= \frac{1 \times 6^3}{12} + \{6 \times 1{\cdot}93^2\}$$

$$= 18 + 22{\cdot}4$$

$$= 40{\cdot}4 \text{ inch units.}$$

Total moment of inertia of section about XX

$$= 118{\cdot}33 + 73{\cdot}83 + 40{\cdot}4$$

$$= 232{\cdot}56 \text{ inch units.}$$

Total moment of inertia of section about YY

$$= \frac{2 \times 8^3}{12} + \frac{6 \times 1^3}{12} + \frac{1 \times 4^3}{12}$$

$$= 85{\cdot}33 + 0{\cdot}5 + 5{\cdot}33$$

$$= 91{\cdot}16 \text{ inch units}$$

54. Moment of Inertia of Symmetrical Beam Sections. When the area is equally distributed about XX as in each of the sections in Fig. 30, the moment of inertia about XX is given as follows—

For 1, 2, and 3 $I_{XX} = \frac{bd^3}{12} - \frac{ah^3}{12}$

for 4 $I_{XX} = \frac{\pi R^4}{4} - \frac{\pi r^4}{4} = I_{YY}$

The moment of inertia of (1) about YY may be found by dividing the section into three rectangles, and adding together the moment of inertia of each rectangle about YY.

The moment of inertia of (2) about $YY = \frac{db^3}{12} - \frac{ha^3}{12}$.

Sections 1 and 2 are each symmetrical about YY, but since 3 is not symmetrical about YY, we proceed as follows—

$$\text{for 3 } I_{YY} = \frac{db^3}{12} + dbr_1{}^2 - \left(\frac{ha^3}{12} + har^2\right)$$

where r is the distance of the centroid of the section from the centroid of the rectangle whose sides are h and a, and r_1 that from the centroid of the larger rectangle.

$$\text{In 5 } I_{XX} = \frac{t_1d^3}{12} + \frac{ht_2{}^3}{12}$$

$$I_{YY} = \left(\frac{dt_1{}^3}{12} + dt_1r_1{}^2\right) + \left(\frac{t_2h^3}{12} + t_2hr_2{}^2\right)$$

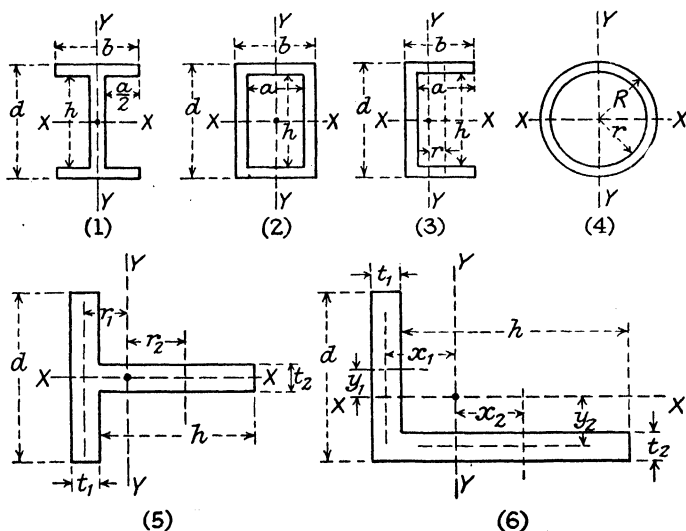



Fig. 30

where r_1 is the distance of the centroid of the section from the centroid of the rectangle $d \times t_1$ and r_2 is the distance of the centroid of the figure from the centroid of the rectangle $h \times t_2$.

$$\text{In 6} \quad I_{XX} = \left\{\frac{t_1d^3}{12} + t_1dy_1{}^2\right\} + \left\{\frac{ht_2{}^3}{12} + ht_2y_2{}^2\right\}$$

$$\text{and } I_{YY} = \left\{\frac{dt_1{}^3}{12} + dt_1x_1{}^2\right\} + \left\{\frac{t_2h^3}{12} + t_2hx_2{}^2\right\}$$

where x_1 and x_2 are the horizontal distances of the centroids of rectangles $d \times t_1$ and $h \times t_2$ from the centroid of the section, and y_1 and y_2 are the corresponding vertical distances.

In the sections 1, 2, 3, 5, and 6, small fillets are found, and the sharp corners of some of the rectangles are slightly rounded. No serious error is introduced, however, by treating the figures as though the corners were sharp.

55. **Ellipse of Inertia.** In a plane figure, if two perpendicular axes are taken, lying in the plane of the figure, and passing

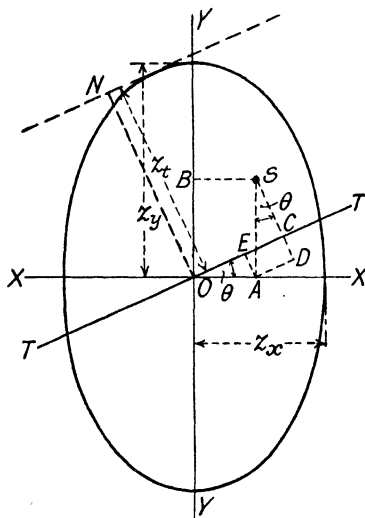



FIG. 31

through its centroid, and if x and y are the rectangular co-ordinates of an element of area a, of the figure, with respect to the axes ; then the value of Σaxy is called the *product of inertia*, or *product moment*.

If the axes are so chosen that the product of inertia is zero, then the axes are called the *principal axes* of the figure. From the definition of product of inertia it follows, if a figure is symmetrical about an axis, then that axis is a principal axis.

Let XX and YY (Fig. 31) be the principal axes of a plane figure, and let z_x and z_y be the radii of gyration of the figure in the directions X and Y respectively.

Let a be a small element of area of the figure at S, whose

rectangular co-ordinates are SA and SB, equal to y and x respectively.

$$I_T = \Sigma\, a\, SC^2 = \Sigma\, a\,(SD - CD)^2 = \Sigma a\,(SD^2 + CD^2 - 2SD\,.\,CD)$$

$$= \Sigma\, ay^2 \cos^2\theta + \Sigma\, ax^2 \sin^2\theta - 2\,\Sigma\, axy \sin\theta\cos\theta \quad (1)$$

$$P_{N.T.} = \Sigma a\,(SC\,.\,OC) = \Sigma a\,(y\cos\theta - x\sin\theta)\,(x\cos\theta + y\sin\theta)$$

$$= \Sigma a\,(y^2 \sin\theta\cos\theta - x^2\sin\theta\cos\theta - xy\sin^2\theta + xy\cos^2\theta) \quad (2)$$

and since X and Y are principal axes.

$$\Sigma\, axy = 0$$

$$\therefore\quad I_T = I_X\cos^2\theta + I_Y\sin^2\theta \quad (3)$$

and $$P_{N.T.} = (I_X - I_Y)\sin\theta\cos\theta \quad (4)$$

also it may be shown that

$$I_N = I_Y\cos^2\theta + I_X\sin^2\theta \quad (5)$$

subtracting (5) from (3) and combining with (4)

$$\text{Tan } 2\,\theta = \frac{2\,P_{N.T.}}{I_T - I_N} \quad (6)$$

by par. 53, $I_X + I_Y = I_T + I_N$

$$\therefore\quad I_X = \tfrac{1}{2}\,\{(I_T + I_N) + (I_T - I_N)\sec 2\,\theta\} \quad (7)$$

$$I_Y = \tfrac{1}{2}\,\{(I_T + I_N) - (I_T - I_N)\sec 2\,\theta\} \quad (8)$$

These equations enable us to calculate the principal moments of inertia when θ is found from (6) and the moment of inertia known about two perpendicular axis ON and OT.

Equations (3), (4) and (6) should be compared with the corresponding equations for stress and strain.

In Fig. 31A Oh and hd represent I_T and $P_{N.T.}$ respectively then $Oa = x = I_X\cos\theta$ and $ad = y = I_Y\sin\theta$

$$\therefore\quad \frac{x^2}{I_X^2} + \frac{y^2}{I_Y^2} = 1$$

This is the equation of an ellipse, and is the locus of d. Such an ellipse is called an *ellipse of inertia*. Thus if we draw two concentric circles of radii I_X and I_Y then draw On at angle θ to I_X, project na and mb on the axes of I_X and I_Y respectively, then Oh is the moment of inertia I_T about an axis inclined at angle θ to the axis of I_X and hd is the product of inertia $P_{N.T.}$ about this axis and a perpendicular axis.

Also if On_1 is drawn perpendicular to On and n_1d_1 drawn parallel to the axis of I_Y, then Oh_1 is equal to I_N the moment of

inertia about an axis inclined at an angle $90 + \theta$ to the axis I_X, and $d_1 h_1$ is the product of inertia $P_{N.T.}$ about the two perpendicular axes.

From triangle dhn $\frac{I_X - I_T}{P_{N.T.}} = \tan\theta$ . . . . . (9)

and from triangle $d_1 h_1 n_1$ $\frac{I_X - I_N}{P_{N.T.}} = \cot\theta$ . . . . (10)

$\therefore \quad (I_X - I_T)(I_X - I_N) = P_{N.T.}^2$ . . . . . (11)

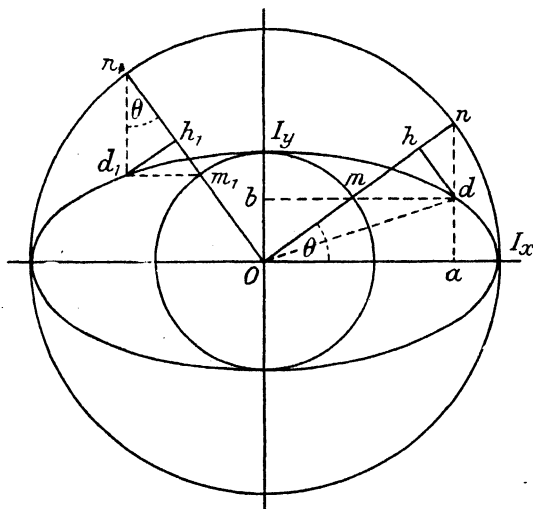



FIG. 31A

The principal moments of inertia and their inclination to OT and ON may also be calculated from (9) and (11)

From (3) $z_t^2 = z_y^2 \cos^2\theta + z_x^2 \sin^2\theta$ . . . . . (12)

Draw an ellipse with major and minor axes z_y and z_x respectively (Fig. 31), erect ON perpendicular to TT and draw a tangent to the ellipse parallel to TT, then by a property of the ellipse,

$$ON^2 = OY^2 \cos^2\theta + OX^2 \sin^2\theta$$
$$= z_y^2 \cos^2\theta + z_x^2 \sin^2\theta$$

$\therefore \quad ON = z_t =$ radius of gyration about TT

This ellipse is called the *momental ellipse*, and is useful in the solution of problem on unsymmetrical bending.

EXAMPLE 5.

A standard unequal angle bar has a section 5 in. × 4 in. × ½ in. Find the greatest and least radius of gyration of the section and the inclination of the principal axes to the sides of the angle.

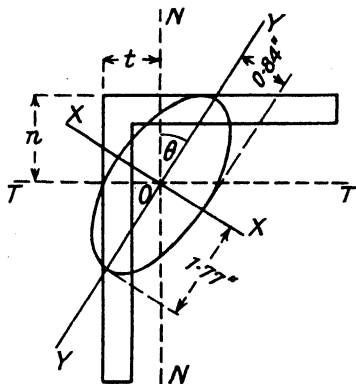



FIG. 31B

Assuming all corners of the section to be square as in Fig. 31B,

$$n = \frac{(3\cdot5 \times 0\cdot5 \times 0\cdot25) + (5 \times 0\cdot5 \times 2\cdot5)}{(3\cdot5 \times 0\cdot5) + (5 \times 0\cdot5)}$$

$$= \frac{6\cdot6875}{4\cdot25}$$

$$= 1\cdot57 \text{ in.}$$

$$t = \frac{(5 \times 0\cdot5 \times 0\cdot25) + (3\cdot5 \times 0\cdot5 \times 2\cdot25)}{(3\cdot5 \times 0\cdot5) + (5 \times 0\cdot5)}$$

$$= \frac{4\cdot565}{4\cdot25}$$

$$= 1\cdot07 \text{ in.}$$

$$I_N = \frac{0\cdot5 \times 3\cdot5^3}{12} + (0\cdot5 \times 3\cdot5 \times 1\cdot18^2) + \frac{5 \times 0\cdot5^3}{12} + (5 \times 0\cdot5 \times 0\cdot82^2)$$

$$= 5\cdot9 \text{ inch units.}$$

$$I_T = \frac{3\cdot5 \times 0\cdot5^3}{12} + (3\cdot5 \times 0\cdot5 \times 1\cdot32^2) + \frac{0\cdot5 \times 5^3}{12} + 0\cdot5 \times 5 \times 0\cdot93^2)$$

$$= 10\cdot4 \text{ inch units.}$$

Product of inertia for ON and OT

$$
\begin{aligned}
&= (2{\cdot}93 \times 0{\cdot}5 \times 1{\cdot}465 \times 1{\cdot}32) \\
&\quad + 1{\cdot}07 \times 0{\cdot}5 \times 1{\cdot}32 \times (-0{\cdot}535) \\
&\quad + 1{\cdot}07 \times 0{\cdot}5 \times 0{\cdot}535 \times (-0{\cdot}82) \\
&\quad + 3{\cdot}43 \times 0{\cdot}5 \times (-1{\cdot}715) \times (-0{\cdot}82) \\
&= 2{\cdot}83 - 0{\cdot}377 - 0{\cdot}234 + 2{\cdot}41 \\
&= 4{\cdot}629
\end{aligned}
$$

$$
\begin{aligned}
\tan 2\theta &= \frac{2 \text{ product of inertia for } ON \text{ and } OT}{I_T - I_N} \\
&= \frac{2 \times 4{\cdot}629}{10{\cdot}4 - 5{\cdot}9} = \frac{9{\cdot}258}{4{\cdot}5} \\
&= 2{\cdot}055 \\
2\theta &= 64^\circ \\
\theta &= 32^\circ
\end{aligned}
$$

The greatest and least moments of inertia are found from—

$$
\begin{aligned}
I_Y &= \tfrac{1}{2}\{(I_T + I_N) - (I_T - I_N)\sec 2\theta\} \\
&= \tfrac{1}{2}\left\{(10{\cdot}4 + 5{\cdot}9) - (10{\cdot}4 - 5{\cdot}9)\frac{1}{0{\cdot}4384}\right\} \\
&= \tfrac{1}{2}\{16{\cdot}3 - 10{\cdot}3\} = 3 \text{ inch units} \\
I_X &= \tfrac{1}{2}\{(I_T + I_N) + (I_T - I_N)\sec 2\theta\} \\
&= \tfrac{1}{2}\{16{\cdot}3 + 10{\cdot}3\} = 13{\cdot}3 \text{ inch units.}
\end{aligned}
$$

$\therefore$ Greatest radius of gyration $= \sqrt{\dfrac{13{\cdot}3}{4{\cdot}25}} = 1{\cdot}77$ in.

Least radius of gyration $= \sqrt{\dfrac{3{\cdot}0}{4{\cdot}25}} = 0{\cdot}84$ in.

The values of I_X I_Y and θ should also be found from (9) and (11), page 84.

EXAMPLES V

1. An H-section beam has an overall depth of 16 in. and each flange is 6 in. $\times$ 0·85 in., the web being 0·55 in. thick. Find the moment of inertia of the section about an axis through the centroid and parallel to the flanges, and also about a perpendicular axis through the centroid.

Ans., 725·9 and 31 in. units.

2. A steel channel has a web ½ in. thick, each flange being 0·6 in. thick, and its outer dimensions being 11 in. × 4 in. Find the moment of inertia about a pair of axes as in Q. 1. *Ans.*, 170·4 and 12·8 in. units.

3. A cast-iron beam section has the dimensions shown by Fig. 32. Find the value of I_{XX}, I_{YY}, and I_{AA}, where XX and YY pass through the centroid of the section. *Ans.*, 591·53 and 162·83 in. units, and 1245 in. units.

4. A stanchion is composed of two H-section beams as shown by Fig. 33, each beam having the dimensions given in Q. 1. Find the radius of gyration of the section of the stanchion about XX and YY : Web centres 12 in. apart. *Ans.*, 6·31 in. and 6·12 in.

5. The section of a stanchion is of the form given in Q. 4. Each beam has an overall measurement of 8 in. × 6 in., web 0·44 in. thick, and flanges 0·56 in. thick. Determine the distance between the centres of the webs in order that the greatest and least radius of gyration may be equal. *Ans.*, 6 in.

6. A plate girder has each flange composed of two plates, one plate being 16 in. × ⅜ in., and the other 16 in. × ½ in., and the web composed of a plate

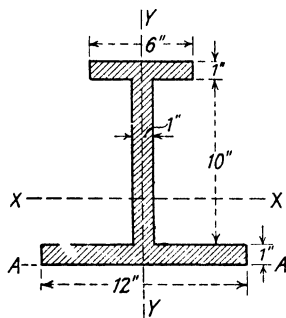



FIG. 32

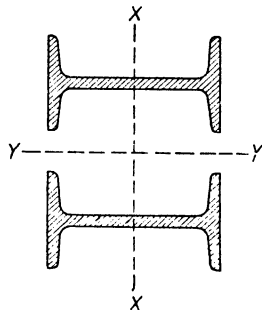



FIG. 33

48 in. × ⅝ in. The web is secured to the flanges, at top and bottom, by two angle bars each 3½ in. × 3½ in. × ½ in. Find the moment of inertia of the section of the girder about an axis through the centroid and parallel to the flanges. *Ans.*, 29,100 in. units.

7. A standard unequal angle bar has a section 6 in. × 4 in. × ½ in. Find its greatest and least radius of gyration, and the inclination of the principal axes to the sides of the angle. *Ans.*, 2·03 in., 0·86 in., 23½°.

8. A rolled T-bar 6 in. × 4 in. × ⅝ in. is to be used as a strut. Find the moment of inertia and radius of gyration of the cross-section about a neutral axis (i.e. axis through centroid) parallel to the top table. (Lond. Univ., 1914.) *Ans.*, 7·35 in. units, 1·12 in.

9. An angle iron 8 in. × 8 in. × 0·55 in. is used as a strut. Determine the principal axes of the section, and the radii of gyration about the principal axes. Draw the ellipse of inertia of the section. (Lond. Univ., 1912.) *Ans.*, 3·12 in. and 1·58 in.

CHAPTER VI

BENDING MOMENTS AND SHEARING FORCES

56. **Definitions.** Consider the horizontal beam, Fig. 34, which is in equilibrium under the forces R_1 and R_2 acting vertically upwards, and the vertical downward forces W_1, W_2, and W_3. The forces keeping the portion, to the right-hand side of a section at A, in equilibrium are $R_2 - W_3$, and the forces exerted by the left-hand portion of the beam across the section at A on the right-hand portion.

Thus force exerted by right-hand portion of beam on left-hand portion $= R_2 - W_3$, and similarly the force exerted by the

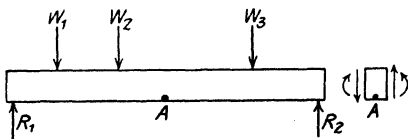



FIG. 34

left-hand portion of beam on right-hand portion $= W_1 + W_2 - R_1$.

A thin longitudinal slice of the beam as at A containing the point A is thus kept in equilibrium by two opposite parallel forces, and since there are no other forces acting on the slice (neglecting its weight), the parallel forces must be equal in magnitude. Thus the slice is in a state of *shear*.

The *shearing force* at any point along a loaded beam is the algebraic sum of all the vertical forces acting to one side of the point.

The forces R_2 and W_3 have a resultant moment acting on the thin slice in an anti-clockwise direction, and the forces W_1, W_2, and R_1 have a resultant moment acting on the slice in a clockwise direction. Since the slice in the beam has no rotary motion, the two resultant moments must be equal in magnitude.

The *bending moment* at any point along a loaded beam is the algebraic sum of the moments of all the vertical forces acting to one side of the point about the point.

57. **Bending Moment and Shearing Force Diagrams.** In order to design a beam to carry a given load system it is necessary to know the value of the bending moment and shearing force at any point along the span. These are usually calculated for various points, and diagrams are then plotted showing the bending moment and shearing force as ordinates, and the span as abscissae. A number of typical cases will now be considered.

(1) Cantilever with concentrated load at free end (Fig. 35A). The bending moment at a point A distant x from the free end is given by

$$M_A = Wx$$

thus the bending moment increases gradually from zero at the free end where $x = O$ to a maximum at the fixed end where $x = L$. The bending moment diagram is therefore a triangle.

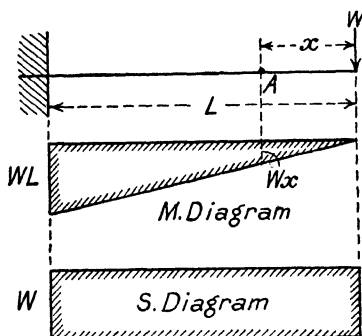



FIG. 35A

The shearing force at $A = W$, hence the shearing force is uniform from end to end of the beam, and the diagram is therefore rectangular.

(2) Cantilever with several concentrated loads (Fig. 35B). The bending moment and shearing force diagram can be drawn separately for W_1, W_2, and W_3, then add together the ordinates of each bending moment diagram to get the bending moment diagram for the combined system, and similarly add together the ordinates of the shearing force diagrams in order to get the shearing force diagram for the system. This follows directly from the definition of bending moment and shearing force.

(3) Cantilever with uniform load w per unit run (Fig. 36A).

The load on the small piece of span of length x, measured from the free end of the cantilever, is wx, and the distance of its centre of gravity from $A = \frac{x}{2}$.

$$\therefore M_A = wx\frac{x}{2} = \frac{wx^2}{2}$$

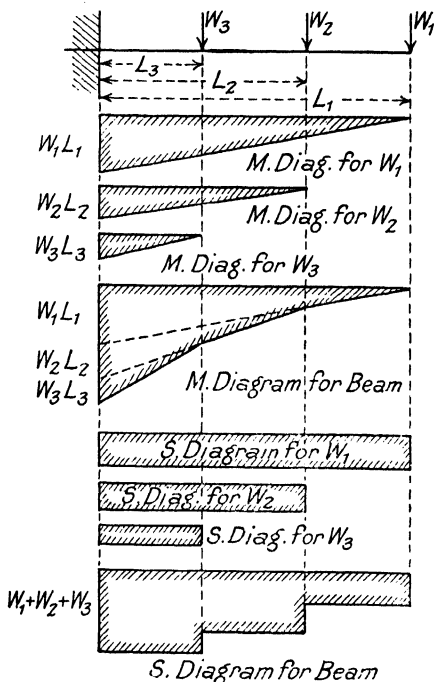



FIG. 35B

Plotting bending moment against x, we obtain a parabola, and the maximum value of M is obtained when $x = L$, and is given by

$$M_{max} = \frac{wL^2}{2} = \frac{WL}{2}$$

Considering the forces to the right-hand side of A, the shearing force at A is wx. The shearing force diagram is therefore triangular, and $S_{max} = wL = W$, and occurs at the constraint.

(4) Cantilever with uniform load w per unit run and several concentrated loads (Fig. 36B).

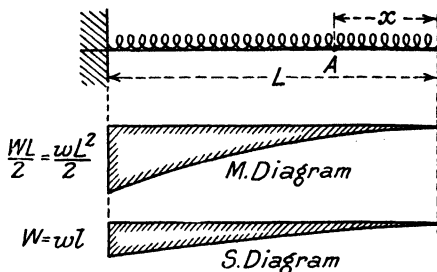



FIG. 36A

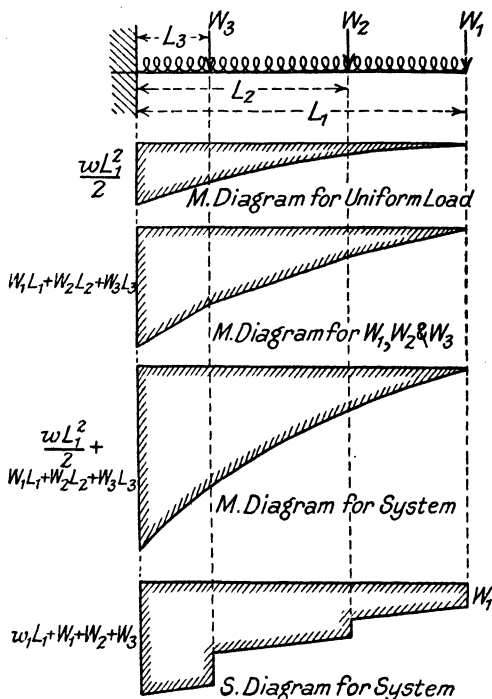



FIG. 36B

The bending moment diagrams for the uniform load, and for the concentrated loads, are obtained as in 35B and 36A, the ordinates of the diagrams added together give the bending

moment diagram for the load system. Similarly, the shearing force diagrams for the uniform load, and the concentrated loads, are drawn, and the ordinates of these diagrams added together give the shearing force diagram for the system.

EXAMPLE 1.

A cantilever 20 ft. long carries loads of 5, 2, and 3 tons at distances of 5, 15, and 20 ft. from its fixed end respectively. Estimate the maximum bending moment and shearing force on the cantilever, and the intensity of a uniformly distributed load which would cause the same bending moment.

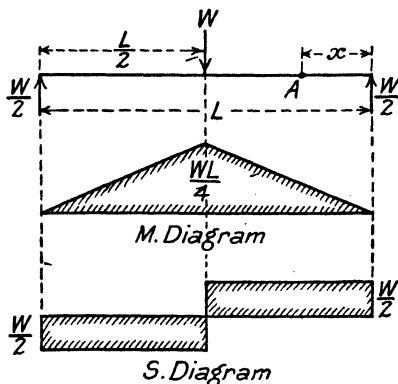



FIG. 37A

The maximum bending moment and shearing force occur at the fixed end.

$$M_{max} = (5 \times 5) + (2 \times 15) + (3 \times 20) = 25 + 30 + 60$$
$$= 115 \text{ ton ft.}$$
$$S_{max} = 5 + 2 + 3$$
$$= 10 \text{ tons.}$$

Let w be the intensity of the uniformly distributed load, which gives the same maximum bending moment,

$$\text{then } M_{max} = \frac{wL^2}{2} = \frac{w}{2} \times 20^2$$

$$\therefore \frac{w}{2} \times 400 = 115$$

$$w = \frac{230}{400}$$

$$= 0{\cdot}575 \text{ ton/ft. run.}$$

(5) Freely supported horizontal beam with concentrated load at mid-span (Fig. 37A).

From symmetry the reaction at each support is $\frac{W}{2}$.

Considering the right-hand half of the span, the bending moment at A, a distance x from the right-hand support is given by

$$M_A = \frac{W}{2} x$$

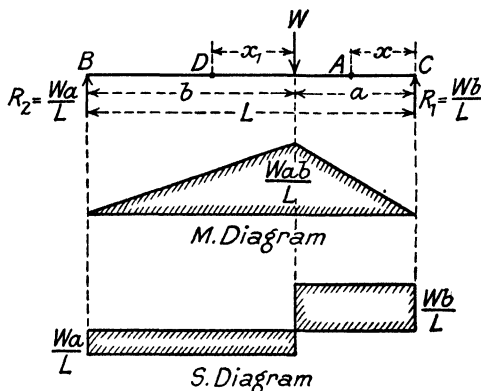



FIG. 37B

The bending moment, therefore, gradually increases from zero at the support, where x is zero, to $\frac{W}{2}\frac{L}{2} = \frac{WL}{4}$ where x is $\frac{L}{2}$.

If we consider the left-hand half of the span, it is found that the bending moment at the left-hand support is zero and increases gradually to $\frac{WL}{4}$ at mid-span. Hence the maximum bending moment occurs at mid-span, and is given by

$$M_{max} = \frac{WL}{4}$$

The shearing force at x is $\frac{W}{2}$ and is uniform over the right-hand half of the span. After passing mid-span the shearing force changes sign, but is still uniform over the left-hand half of the span and equal to $\frac{W}{2} - W = -\frac{W}{2}$.

(6) Freely supported beam with concentrated load not at mid-span (Fig. 37B).

Let the load divide the span into two portions a and b. Since the load is not at mid-span, the reaction at each end will be different. Taking moments about B, we have

$$R_1 L = Wb$$

$$\therefore R_1 = \frac{Wb}{L}$$

$$\text{and } R_1 + R_2 = W$$

$$\therefore R_2 = W - R_1$$

$$R_2 = W - \frac{Wb}{L} = W\left(\frac{L - b}{L}\right)$$

$$\therefore R_2 = \frac{Wa}{L}$$

Considering the right-hand portion of the span, the bending moment at A is given by

$$M_A = R_1 x = \frac{Wb}{L} x$$

Thus the bending moment diagram for the right-hand half of the span will be triangular, increasing from zero at $x = 0$ to $\frac{Wb}{L} a$ where $x = a$.

The bending moment at D, a distance x_1 to the *left* of W, is given by

$$M_D = R_1 (a + x_1) - Wx_1$$

$$= \frac{Wb}{L} (a + x_1) - Wx_1$$

$$= W\left\{\frac{ab}{L} + x_1\left(\frac{b}{L} - 1\right)\right\}$$

$$= W\left\{\frac{ab}{L} + x_1\left(\frac{b - L}{L}\right)\right\}$$

$$= W\left\{\frac{ab}{L} - x_1\frac{a}{L}\right\}$$

This gives a triangular bending moment diagram for the left-hand portion of the span, the bending moment decreasing from $\frac{Wab}{L}$ where x_1 is zero to

$$M_B = W\left(\frac{ab}{L} - \frac{ab}{L}\right) = 0$$

where $x_1 = b$. The maximum bending moment is thus given by

$$M_{max} = \frac{Wab}{L}$$

The shearing force over the right-hand portion of the span is uniform and equal to $R_1 = \frac{Wb}{L}$. After passing W the shearing force becomes $\frac{Wb}{L} - W = -\frac{Wa}{L}$, and is uniform over the left-hand portion of the span.

(7) Horizontal freely supported beam with concentrated loads and overhanging end.

The dimensions and load distribution are given by Fig. 38.

$$20R_2 + 20 = 6 + 60 + 54 = 120$$

$$\therefore R_2 = \frac{100}{20} = 5 \text{ tons}$$

$$R_1 + R_2 = 3 + 5 + 2 + 4 = 14$$

$$\therefore R_1 = 14 - 5 = 9 \text{ tons}$$

$M_F = 0$

$M_E = 4 \times 5 = 20$ ton ft.

$M_D = (4 \times 8) - (9 \times 3) = 32 - 27 = 5$ ton ft.

$M_C = (4 \times 17) - (9 \times 12) + (2 \times 9) = 68 - 108 + 18 = 86 - 108 = -22$ ton ft.

$M_B = (4 \times 23) - (9 \times 18) + (2 \times 15) + (5 \times 6) = 92 - 162 + 30 + 30 = 152 - 162 = -10$ ton ft.

$M_A = 0$

$S_{F-E} = 4$ tons

$S_{E-D} = 4 - 9 = -5$ tons

$S_{D-C} = 4 - 9 + 2 = -3$ tons

$S_{C-B} = 4 - 9 + 2 + 5 = 2$ tons

$S_{B-A} = 4 - 9 + 2 + 5 + 3 = 5$ tons.

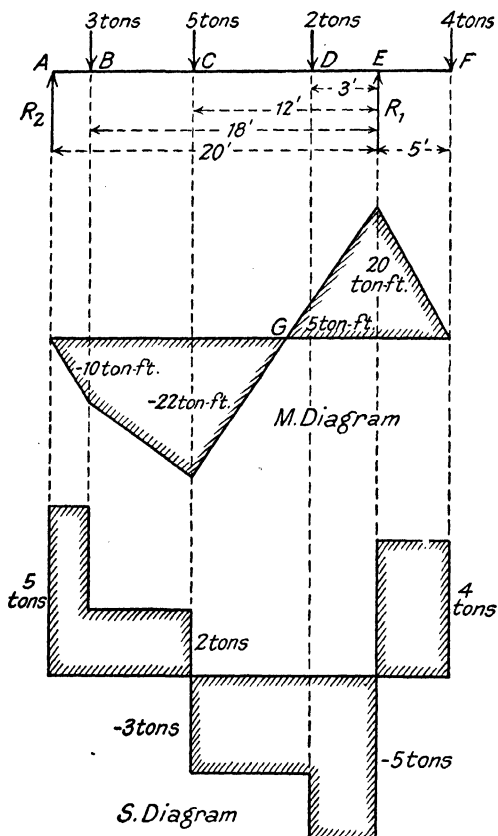



Fig. 28

The bending moment and shearing force diagrams are shown in Fig. 38.

The point G on the beam where the bending moment is zero, and, as shall be found later where the curvature changes sign, is called a *point of inflection* or *point of contraflexure.*

(8) Horizontally supported beam with equal overhanging ends at which isolated loads act.

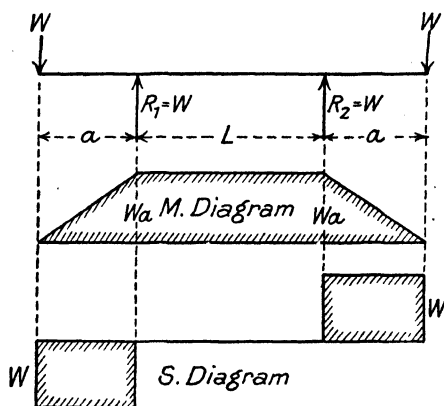



FIG. 39

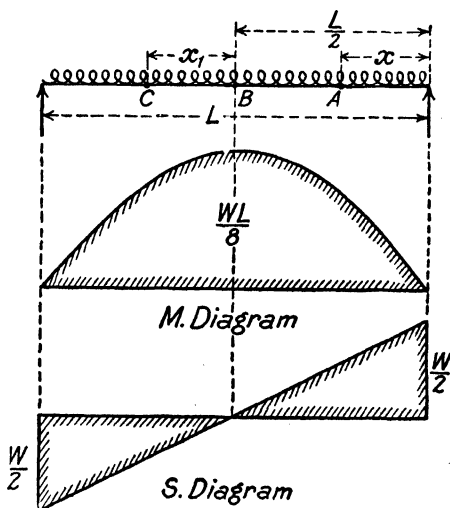



FIG. 39A

An important example of this case is a wagon axle. The supports or wheels are R_1 and R_2, and the load at each bearing is W.

The diagrams are shown by Fig. 39, the bending moment

being uniform between the supports or wheels, and the shearing force being zero between the same points.

$$M_{max} = Wa$$

$$S_{max} = W$$

(9) Horizontally supported beam with uniform load w per unit run (Fig. 39A).

The total load on the beam is $wL = W$, and from symmetry the reaction at each end is $\frac{W}{2}$.

$M_A = \frac{W}{2}x - wx\frac{x}{2} = \frac{W}{2}x - \frac{wx^2}{2}$. Thus the bending moment diagram is parabolic in form, increasing from zero where $x = 0$ to a value $\frac{W}{2}\frac{L}{2} - \frac{wL^2}{8} = \frac{WL}{4} - \frac{WL}{8} = \frac{WL}{8}$ where $x = \frac{L}{2}$.

$M_C = \frac{W}{2}\left(\frac{L}{2} + x_1\right) - \frac{w}{2}\left(\frac{L}{2} + x_1\right)^2$. This expression also gives a parabolic bending moment diagram for the left-hand side of the diagram, decreasing from $\frac{WL}{8}$ where x_1 is zero, to zero where x_1 has the value $\frac{L}{2}$.

$\therefore M_{max} = \frac{WL}{8}$ and occurs at mid-span.

$S_A = \frac{W}{2} - wx$. The shearing force diagram is thus triangular for the right-hand side of the span, decreasing from $\frac{W}{2}$ where $x = 0$ to zero at mid-span where $x = \frac{L}{2}$.

$S_C = \frac{W}{2} - w\left(\frac{L}{2} + x_1\right)$. This gives a triangular diagram for the left-hand side of the diagram, the shearing force increasing from zero where $x_1 = 0$ to $-\frac{W}{2}$ where $x_1 = \frac{L}{2}$.

$\therefore S_{max} = \frac{W}{2}$ at each end of the span.

(10) Horizontally supported beam with equal overhanging ends, and a uniform load w per unit run, over the beam.

Each overhanging end acts as a cantilever, and the maximum bending moment at the supports is $\frac{wa^2}{2}$. If the load was removed from b the bending moment diagram would be $PQRT$.

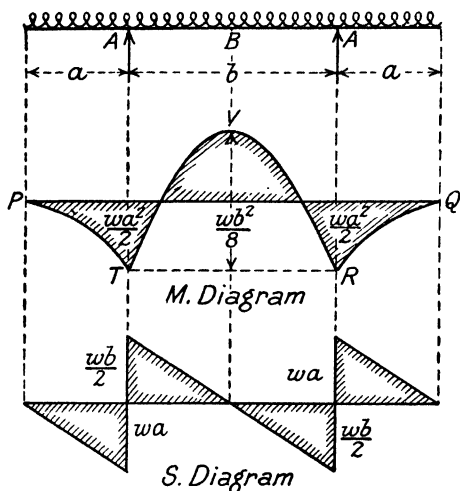



FIG. 40

The diagram for the portion of span b is a parabola whose maximum ordinate is $\frac{wb^2}{8}$. This bending moment, due to the load on b, is of opposite sign to that due to the two end loads; hence, as the diagram due to b is drawn on TR as base, we obtain the resultant diagram shown by Fig. 40.

The value of $M_B = \frac{wb^2}{8} - \frac{wa^2}{2}$

" " $M_A = \frac{wa^2}{2}$

The resultant bending moment at the centre will be zero when $M_B = 0$,

$$\text{i.e. } \frac{wa^2}{2} = \frac{wb^2}{8}$$

$$\text{or } a = \frac{b}{2}$$

When a increases, M_A increases and M_B decreases. Thus when the bending moment at mid-span is zero, the bending moment at each support has the value $\frac{wb^2}{8}$; the point V would then just touch the line PQ.

The least bending moment on the beam is obtained when M_A is equal to M_B,

or $$\frac{wa^2}{2} = \frac{wb^2}{8} - \frac{wa^2}{2}$$

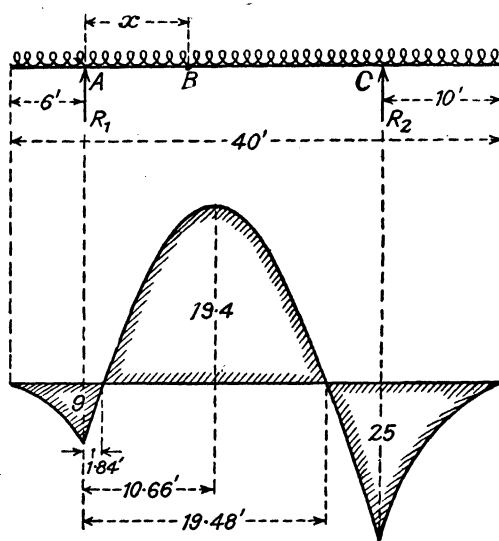
$$\therefore a = \frac{b}{2\sqrt{2}}$$

or $a \doteqdot 0{\cdot}354b$

M. Diagram

FIG. 40A

EXAMPLE 2.

A beam 40 ft. long, resting on two supports 24 ft. apart, centre to centre, projects to the extent of 6 ft. and 10 ft. beyond the supports at either end, and it is loaded uniformly with 0·5 ton per foot. Find the position and magnitude of the greatest bending moment on the beam and the points of contraflexure. Make a dimensioned sketch bending moment diagram for the beam. (A.M.I.Mech.E., 1924.)

Let R_1 and R_2 be the reactions of the supports (Fig. 40A).

Then $$R_1 \times 24 = 40 \times 0{\cdot}5 \times 10$$

$$R_1 = \frac{40 \times 5}{24} = 8\tfrac{1}{3} \text{ tons}$$

The bending moment at $A = -6 \times 0{\cdot}5 \times 3 = -9$ ton ft.

" " " B distant x from A

$$= 8\tfrac{1}{3} \times x - (6 + x)\, 0{\cdot}5 \left(\frac{6 + x}{2}\right)$$

$$= 8\tfrac{1}{3}x - \tfrac{1}{4}(36 + 12x + x^2)$$

$$= -\tfrac{1}{4}x^2 + 5\tfrac{1}{3}x - 9$$

For maximum bending moment between supports $\frac{dM}{dx} = 0$

$$\therefore -\tfrac{1}{2}x + 5\tfrac{1}{3} = 0$$
$$\text{or } x = 10\tfrac{2}{3} \text{ ft.}$$
$$\therefore \quad M_{max} = 19{\cdot}4 \text{ ton ft.}$$

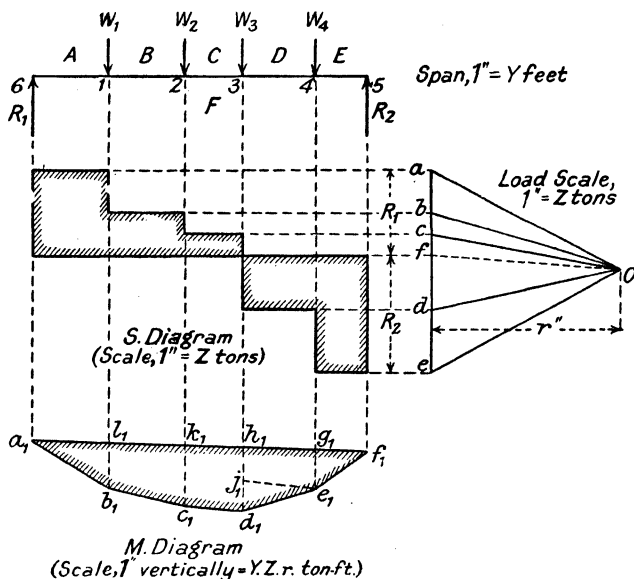



FIG. 41

At points of contra flexure $M = 0$,

$$\therefore \tfrac{1}{4}x^2 - 5\tfrac{1}{3}\,x + 9 = 0$$

$$x = \frac{5\tfrac{1}{3} \pm \sqrt{(5\tfrac{1}{3})^2 - 9}}{\tfrac{1}{2}} = (5\tfrac{1}{3} \pm 4{\cdot}41)2$$

$$= 1{\cdot}84 \text{ and } 19{\cdot}48 \text{ ft.}$$

$$M_C = 10 \times 0{\cdot}5 \times 5$$

$$= 25 \text{ ton ft. and is greatest on the beam.}$$

58. **Bending Moments, by Graphical Methods.** A beam freely supported at each end and carrying concentrated loads is shown by Fig. 41. The spaces between the loads are lettered A, B, C, D, and E, that between the two reactions R_1, and R_2 being F. Choose a load scale and set down $ab = W_1$. $bc = W_2$,

$cd = W_3$, and $de = W_4$. Take a pole o in any position, and join oa, ob, oc, od, and oe. Draw vertical projectors through the line of action of the reactions and loads. Take a point a_1 on the projector through R_1 and draw a line parallel to oa to cut the projector through W_1 in b_1, draw b_1c_1 parallel to ob, c_1d_1 parallel to oc, d_1e_1 parallel to od, and e_1f_1 parallel to oe. Join a_1f_1 and through o draw of parallel to a_1f_1. af on the load scale is equal to R_1 and fe to R_2. A horizontal line through f can now be used as a base line, and the shearing force diagram is obtained, to the same scale as the load scale, by horizontal projection from a, b, c, d, and e.

The diagram $a_1b_1c_1d_1f_1$ is the bending moment diagram for the beam, a vertical ordinate at any point representing the bending moment, at the point, to scale.

Proof. The triangles $f_1g_1e_1$ and ofe are similar.

$$\therefore \frac{g_1e_1}{f_1g_1} = \frac{fe}{of} = \frac{R_2}{of}$$

$$g_1e_1 = R_2\frac{f_1g_1}{of} = R_2\frac{4\,.\,5\,.}{r}$$

$$= \frac{M_4}{r}$$

Through e_1 draw e_1j_1 parallel to a_1f_1.

The triangles $e_1j_1d_1$ and ofd are similar.

$$\therefore \frac{j_1d_1}{j_1e_1} = \frac{fd}{of}$$

$$j_1d_1 = fd\frac{j_1e_1}{of} = (R_2 - W_4)\frac{3\,.\,4\,.}{r}$$

$$\text{and } h_1d_1 = j_1d_1 + h_1j_1$$

$$= (R_2 - W_4)\frac{3\,.\,4\,.}{r} + R_2\frac{4\,.\,5\,.}{r}$$

$$= \frac{R_2(3\,.\,4\,. + 4\,.\,5\,.) - W_4\,.\,3\,.\,4\,.}{r}$$

$$= \frac{R_2\,.\,(3\,.\,5\,.) - W_4\,.\,(3\,.\,4\,.)}{r}$$

$$= \frac{M_3}{r}$$

Similarly it may be proved that $k_1c_1 = M_2$ and $l_1b_1 = M_1$.

If the beam is set out to a scale of 1 in. = Y feet (say) and the load scale is 1 in. = Z tons, and the horizontal distance of 0 from the load scale = r inches, then on the bending moment diagram 1 in. vertically = $Y \,.\, Z \,.\, r$ ton ft.

Obviously for ease in calculation r should be an integer. If some of the loading is uniformly distributed over the span, a

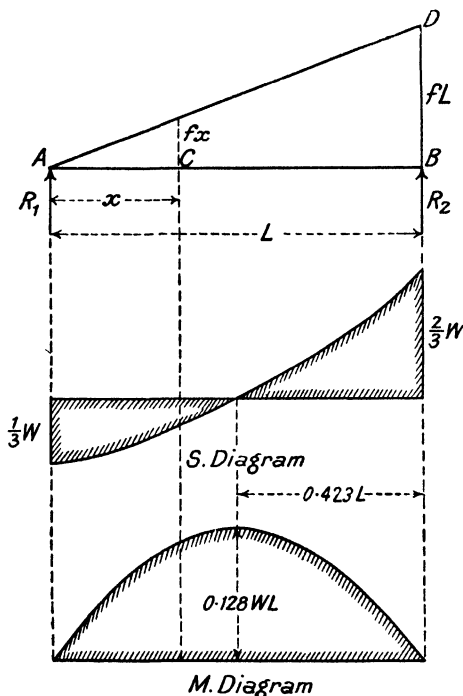



FIG. 42

fair approximation is obtained to the bending moment diagram by dividing the uniform load into small pieces, and treating each piece as a concentrated load acting at its centre of gravity.

59. **Freely Supported Beam with Gradually Increasing Load.** The beam AB carries a load of gradually increasing intensity as represented by the triangular diagram ABD (Fig. 42). If f is the intensity per unit run at unit distance from A then the intensity at c is fx, and that at B, fL.

The total load on AB = average intensity × span

$$= \frac{fL}{2} \times L$$

$$= \frac{fL^2}{2}$$

The centroid of this load is at a horizontal distance $\frac{L}{3}$ from DB, hence, taking moments about B, we have

$$R_1 L = \frac{fL^2}{2} \times \frac{L}{3}$$

$$\therefore R_1 = \frac{fL^2}{6} = \frac{W}{3} \text{ where } W = \text{total load}$$

$$\text{and } R_2 = \frac{2}{3} W$$

The shearing force at $C = S_c = R_1 - \frac{fx^2}{2}$

$$= \frac{fL^2}{6} - \frac{fx^2}{2} \quad . \quad . \quad . \quad (1)$$

and hence the shearing force diagram is parabolic in form.

Since (1) decreases as x increases, the shearing force will be zero when

$$\frac{fL^2}{6} - \frac{fx^2}{2} = 0$$

$$\text{or } x = \frac{L}{\sqrt{3}} = 0{\cdot}577\,L \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The bending moment at C is obtained from

$$M_C = R_1 x - \frac{fx^2}{2} \times \frac{x}{3}$$

$$= \frac{fL^2}{6} x - \frac{fx^3}{6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, therefore, the bending moment diagram is also of parabolic form, being zero when x is zero, and also being zero when

$x = L$. The bending moment will be maximum when the slope of the diagram is zero, i.e. when

$$\frac{dM}{dx} = \frac{fL^2}{6} - \frac{fx^2}{2} = 0$$

$$\text{or } x = \frac{L}{\sqrt{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the bending moment is maximum when the shearing force is zero. It will be seen in a later paragraph that this is true generally.

$$M_{max} = \frac{fL^2}{6} \times \frac{L}{\sqrt{3}} - \frac{fL^3}{3\sqrt{3} \times 6}$$

$$= \frac{fL^3}{6\sqrt{3}}\left(1 - \frac{1}{3}\right)$$

$$= \frac{fL^3}{9\sqrt{3}}$$

$$= \frac{2WL}{9\sqrt{3}}$$

$$= 0{\cdot}128WL \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This problem is of importance in the design of ships' bulkheads, and other structures which have to resist the horizontal thrust exerted by water.

EXAMPLE 3.

A beam freely supported in a horizontal position has a span of 10 ft., and carries a total load of 2 tons. Find the maximum bending moment and shearing force on the beam (*a*) if the load is concentrated at mid-span, (*b*) if the load is uniformly distributed over the span, (*c*) if the intensity of loading increases gradually from zero at one end of the span.

$$(a)\ M_{max} = \frac{WL}{4} = \frac{2 \times 10}{4} = 5 \text{ ton ft.}$$

$$S_{max} = \frac{W}{2} = 1 \text{ ton}$$

$$(b)\ M_{max} = \frac{WL}{8} = \frac{2 \times 10}{8} = 2\tfrac{1}{2} \text{ ton ft.}$$

$$S_{max} = \frac{W}{2} = 1 \text{ ton}$$

(c) $M_{max} = 0{\cdot}128 WL$

$= 0{\cdot}128 \times 2 \times 10$

$= 2{\cdot}56$ ton ft.

$S_{max} = \frac{2}{3} W$

$= \frac{2 \times 2}{3}$

$= 1\frac{1}{3}$ tons

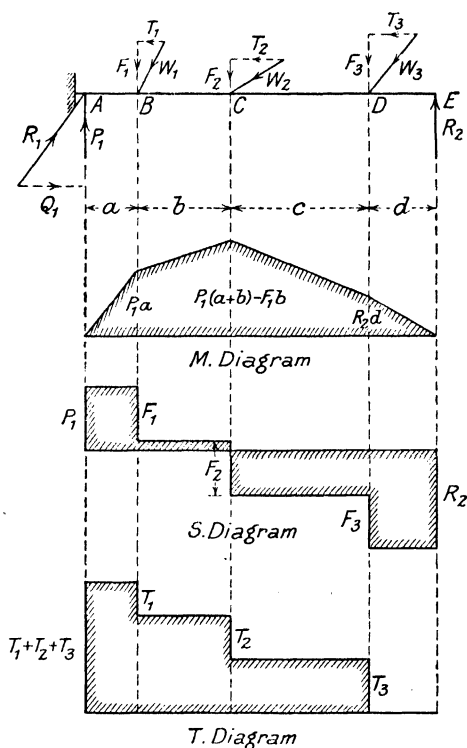



Fig. 43

60. **Freely Supported Beam with Inclined Loads.** The beam AE in Fig. 43 has the inclined loads W_1, W_2, and W_3. It is supported at E and A. The loads can each be resolved into components F perpendicular to the beam, and thrust T parallel to the beam. In order to prevent lateral motion a constraint

must be applied, as shown at A. The reaction at A is R_1, and it can also be resolved into P_1 perpendicular to the beam, and Q_1 parallel to the beam. P_1 can be found by taking moments about E of the forces F_1, F_2 and F_3. The forces F_1, F_2 and F_3 cause shear and bending, and the forces T_1, T_2 and T_3 cause thrust. Diagrams of bending moment, shearing force and thrust are shown.

61. **Shearing Force at Concentrated Loads.** It will be noticed in each of the shearing force diagrams that, where a concentrated load occurs, the diagram changes abruptly, the change

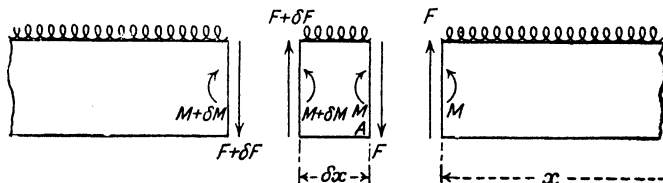



FIG. 44

in shear being denoted by a vertical line and the corner of the diagram being right-angled. In practice it is not possible to make the load act at a *point* on the beam. It is usually spread over a small piece of the span. The diagram, therefore, strictly speaking, should not change so abruptly; the line denoting the change should be slightly inclined to the perpendicular and the corners of the diagram should possess slight rounding.

62. **Relation between** M, S, **and** w. The beam in Fig. 44 of span L carries a load such that w is the average load intensity over the small length of the span δx at a distance x from one end of the beam.

Let the bending moment at x be M, and the bending moment at $x + \delta x$ be $M + \delta M$. The shearing force at x is F and that at $x + \delta x$ is $F + \delta F$. Considering the small piece of the beam of length δx, for equilibrium the bending moments acting on it must be M and $M + \delta M$ and the shearing forces F and $F + \delta F$.

Since all the vertical forces are together zero

$$(F + \delta F) - w\delta x - F = 0$$

$$\text{or } \frac{\delta F}{\delta x} = w$$

$$\text{and } \therefore \frac{dF}{dx} = w \quad . \quad . \quad . \quad . \quad (1)$$

Integrating between o and x we have that the change in shearing force is given by—

$$F_x = \int_0^x dF = \int_0^x w\,dx \qquad . \quad . \quad . \quad . \quad . \quad (2)$$

but $\int_0^x wdx$ is the area of the load diagram over the span x, and hence the change in shearing force between two points on the span is equal to the area of the load distribution diagram for that piece of the span.

Taking moments about A, we have for equilibrium of the piece of length δx that

$$(M + \delta M) - (F + \delta F)\,\delta x + w\delta x \,.\, \frac{\delta x}{2} - M = 0$$

or
$$\delta M - F\delta x = 0$$

if we neglect the product of small quantities.

$$\therefore \frac{\delta M}{\delta x} = F$$

or
$$\frac{dM}{dx} = F \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating between the limits o and x, we have that the change in bending moment between these two points is given by

$$M_x = \int_0^x dM = \int_0^x F dx \qquad . \quad . \quad . \quad . \quad . \quad (4)$$

but $\int_0^x Fdx$ is the area of the shearing force diagram between the limits o and x. Hence the change in bending moment between two points on the span is equal to the area of the shearing force diagram over that portion of the span.

From (3) it is evident that the ordinates of the F diagram are proportional to the slope of the M diagram. Hence, when the slope of the M diagram is zero, i.e. the value of M is a mathematical maximum or minimum, the value of F is zero. This enables us to solve many difficult problems on beams. See also par. 59.

The two rules given above may be stated concisely as follows—

(1) The shearing force curve is the sum curve of the load intensity curve.

(2) The bending moment curve is the sum curve of the shearing force curve.

63. Application of the Sum Curve Rule to Ships. If a body, such as a beam of wood of constant cross-section, floats in water, the upward thrust of the water will, for all portions of the length, equal the downward thrust due to the weight, and consequently there will be no shearing force or bending moment on the body. In the case of a ship the load intensity is not constant, and hence the buoyancy is not constant and equal to the load for all portions of the ship. The difference between the two quantities gives a force which acts sometimes upwards

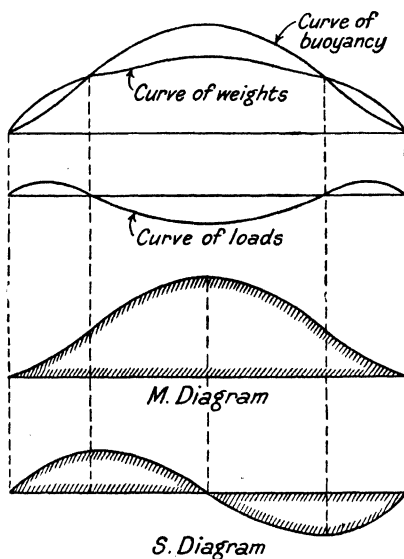



FIG. 45

and sometimes downwards, depending on which is the greater; giving a shear and bending action on the portion under consideration.

The ship is divided along its length into equal parts, and the weight of each part calculated; the average intensity of weight for each part is plotted against length, and the curve of weights is obtained (Fig. 45). The volume of water displaced by each section, in smooth water, is then estimated, and from this the upward thrust of the water per unit run calculated. The intensity of upward thrust is plotted against length, and the curve of buoyancy for smooth water obtained.

The area of the curve of weights is then equal to the area of the curve of buoyancy, since the total weight of the ship is equal to the total upward thrust. The difference between the ordinates of the curves gives the amount the weight exceeds the buoyancy, or vice versa. The difference between the ordinates, plotted to a base of length gives the load intensity curve for the ship in smooth water. The points where the two ordinates are equal, or the load intensity curve is zero, are called " water-borne " sections.

The shearing force and bending moment diagrams are then plotted by the sum curve rule.

The problem is also complicated by the fact that during rough weather the ship may be resting on the crest of a wave, or be supported on the crests of two waves. The former causes " hogging strains " and the latter " sagging strains." For a complete investigation of the problem the reader is referred to works on naval architecture.

EXAMPLES VI

1. The teeth on a spur wheel are 2 in. high and the greatest load acting at the point of the tooth is 300 lb. What is the maximum bending moment which the tooth has to resist ? *Ans.*, 600 lb. in.

2. A solid steel propeller shaft is 12 in. diameter, and is supported at two points 10 ft. apart. Find the maximum bending moment on the shaft due to its own weight. 1 cub. in. of steel weighs 0·29 lb. *Ans.*, 59,007 ib. in.

3. A beam of 12 ft. span carries a load of ½ ton per foot run. What load concentrated at mid-span would give the same bending moment ?
Ans., 3 tons

4. A beam supported at each end has a span of 30 ft., and carries loads of 2, 3, 1, and 6 tons at distances of 5, 10, 20, and 25 ft. from the left-hand support. Calculate the maximum bending moment and shearing force on the beam. *Ans.*, M = 40 ton ft., S = 7 tons.

5. In the previous example find the maximum bending moment and shearing force if the beam is increased in length, so that there is an overhang of 5 ft. at each support, and a load of 2 tons at the end of the left-hand overhang, and a load of ½ ton per foot run spread over the right-hand overhang.
Ans., M = 32·5 ton ft., S = 6·875 tons.

6. A wheel weighing 60 lb. rotates at a uniform speed of 300 r.p.m., and is keyed to a shaft which projects 6 in. beyond a bearing. If the centre of gravity of the wheel is ¼ in. from the axis of rotation, find the greatest bending moment on the shaft. *Ans.*, 594 lb. in.

7. A vertical beam is used to stiffen a bulkhead 30 ft. deep ; the width of plate supported by the beam being 30 in. If water reaches to the top of the bulkhead, calculate the greatest shearing force and bending moment on the stiffener. *Ans.*, 20·86 tons, 120·8 ton ft.

8. The intensity of loading on a beam of 18 ft. span increases gradually from 500 lb./ft. run at one end, to 1,000 lb./ft. run at the other end. Find the point of maximum bending moment and the magnitude of this moment.

Ans., 9·48 ft. from L.H. support, 30,880 lb. ft.

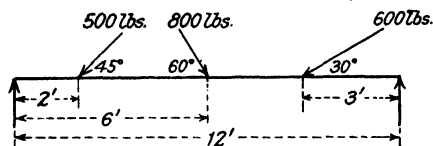



FIG. 46

9. A beam carries load as shown by Fig. 46. Estimate the maximum bending moment, shearing force, and thrust on the beam.

Ans., M = 2,882 lb. ft., S = 715 lb., T = 473 lb.

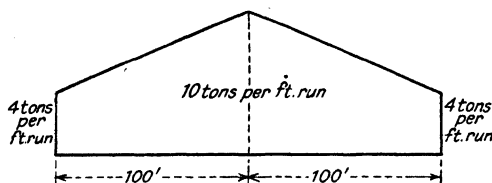



FIG. 47

10. A vessel 200 ft. in length has a buoyancy curve which is a straight line and having a constant ordinate of 7 ton/ft. run. The weight curve is shown in Fig. 47. Draw the bending moment and shearing force curves for the vessel.

11. A beam covers a clear span of 12 ft., and is freely supported at the ends. The beam carries a load of $1\frac{1}{2}$ tons, uniformly distributed on half the span, in addition to a load of $\frac{1}{2}$ ton concentrated at the centre of the span. Calculate the bending moment at the centre of the span, and also the position and value of the maximum bending moment. Draw the shearing force and bending moment diagrams. (Lond. Univ., 1919.)

Ans., 3·75 ton ft., $\frac{1}{2}$ ft. from mid-span, in distributed load ; 3·78 ton ft.

12. A beam carrying a uniformly distributed load rests on two supports b ft. apart, with an equal overhang a ft. at each end. Determine the ratio b/a for zero bending moment at mid-span, also the ratio if the maximum bending moment is as small as possible. What is the most economical length for a railway sleeper if the rail centres are 63 in. apart?

Ans., 2, $2\sqrt{2}$, 9 ft.

CHAPTER VII

STRESSES IN BEAMS

64. **Neutral Axis.** The bending moment and shearing force, at various points along a loaded beam, introduce stresses in the beam, and, with certain assumptions, the connection between the stresses, the bending moment, the curvature of

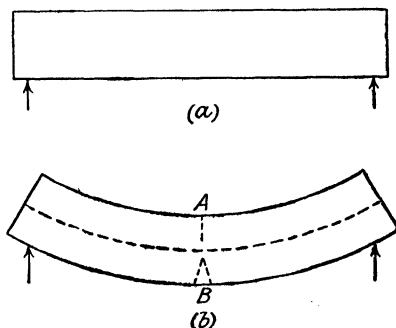



Fig. 48

the beam, the dimensions of the beam and the elasticity of the beam can be obtained. The nature of these stresses may be studied by reference to Fig. 48. A beam of uniform section is shown resting, in the unloaded state, at (*a*). When under a given load system the shape of the beam, greatly exaggerated, is given by (*b*). If we suppose the beam to be of wood, and attempt to make a saw-cut in the direction A to B, it is a matter of common experience that the beam will eventually close in on the saw, thus showing that a state of compression exists at one side of the beam. If the saw-cut is made in the direction B to A, it is also well known that the cut opens out rapidly, thus showing that the other side of the beam is in a state of tension. The fibres at one side of the beam will be shortened, as shown by (*b*), owing to the state of compression existing, and those at the other side will be lengthened owing to the state of tension existing. At a point between the top and bottom of the beam a layer of fibres will be found which suffer no stress, and consequently remain their original length.

This layer of fibres forms what is known as the *neutral surface*, and the trace of this surface on a cross-section (N.A.) is called the *neutral axis*.

65. **Assumptions in the Simple Bending Theory.** When a beam is bent, due to the application of a constant bending moment, i.e. by couples applied to its ends, without being subjected to shear, it is said to be in a state of *simple bending*. In this case the relationship existing between the stresses, etc.,

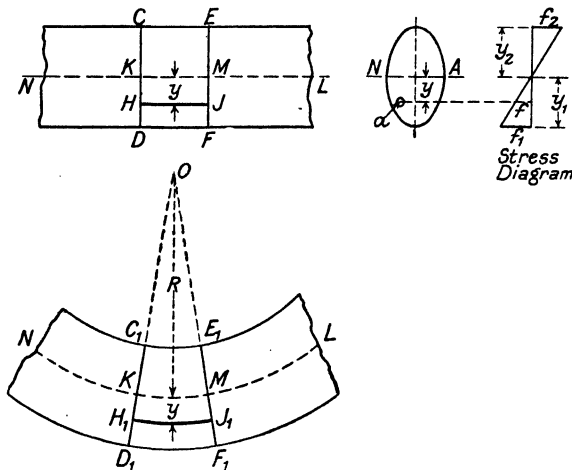



FIG. 49

mentioned in par. 64, are obtained readily with the aid of the following assumptions—

(1) That Young's modulus has the same value, for the material of the beam, in tension as in compression, and that the stress is proportonal to the strain.

(2) That a transverse section of the beam, which is a plane before bending, will remain a plane after bending.

(3) That the radius of curvature of the beam, before bending, is very large in comparison to the transverse dimensions of the beam.

(4) That the resultant pull or thrust across a transverse section of the beam is zero.

(5) That the transverse section of the beam is symmetrical about an axis, passing through the centroid of the section, and parallel to the plane of bending.

In Fig. 49 a portion of a uniform beam, subjected to simple

bending, is shown. In the unstrained state let HJ be a portion of a fibre at the distance y from the neutral surface ; its length being determined by the two parallel planes CD and EF. After bending, the planes assume the position shown, being inclined at the angle θ and intersecting at the point O. Let R be the radius of the neutral surface, the radius of the curved fibre is $(R + y)$. Since the fibre is not at the neutral surface, its length is altered to H_1J_1.

Now $$\frac{H_1J_1}{KM} = \frac{(R + y)\theta}{R\theta} = \frac{R + y}{R}$$

Also the strain in the fibre is given by

$$\frac{H_1J_1 - HJ}{HJ} = \frac{H_1J_1 - KM}{KM} = \frac{H_1J_1}{KM} - 1$$

$$= \frac{y}{R}$$

If f is the intensity of the stress in the fibre then

$$\frac{f}{\frac{y}{R}} = E$$

or $$\frac{f}{y} = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

this may also be expressed in the form $f = \frac{E}{R}y$, and for a given beam under a given load system we may write $f = ky$, or, the stress in the fibres of a beam, at a distance y from the neutral surface, is directly proportional to the distance of the fibre from the neutral surface.

Since the material in proximity to the neutral surface carried no stress, and, therefore, lends no assistance to resist the applied couple, a beam should be such that the greatest possible amount of its area is as far away from the neutral surface as possible. We see this exemplified in the well-known sections adopted in engineering practice, such as H, T, and channel sections, etc.

From (1) it will be observed that, since the beam is of uniform section, R is constant and hence NL will form an arc of a circle.

66. Position of the Neutral Axis. Consider an element of area a at the distance y from the neutral axis. We have that the total force on the element $= f \times a$

but
$$\frac{f}{y} = \frac{f_1}{y_1}$$

$\therefore$ Total force on element $= a \frac{f_1}{y_1} y = \frac{f_1}{y_1} ay$

and total force on transverse section below N.A.

$$= \Sigma \frac{f_1}{y_1} ay$$

$$= \frac{f_1}{y_1} \Sigma ay$$

Similarly, it may be shown that if a and y be chosen on the upper side of N.A., the total force on the transverse section above N.A. $= \frac{f_2}{y_2} \Sigma ay$.

From assumptions (1) and (2), par. 65, we have that

$$\frac{f_1}{y_1} = \frac{f_2}{y_2}$$

Also $\frac{f_1}{y_1} \Sigma ay$ for the area below N.A. is the total tension on the section, and $\frac{f_2}{y_2} \Sigma ay$ for the area above N.A. is the total compression on the section. By assumption (4), par. 65, the resultant of these is zero. $\therefore \Sigma ay$ for the area below N.A. $= \Sigma ay$ for the area above N.A. and the signs are different, hence the value of Σay for the whole section (or the first moment of the section) about N.A. is zero. From par. 50 we see that for this to occur, N.A. must pass through the centroid of the section. We have, therefore, the important rule that, in cases of simple bending, the neutral axis passes through the centroid of the section.

67. Moment of Resistance. The moment of the force, on the small element a, about N.A.

$$= \frac{f_1}{y_1} ay^2$$

and the total moment of all the forces, acting on the various small elements composing the cross-section, forms a couple which is equal to the bending moment. This total moment is called the *Moment of Resistance* = M.

$$\therefore M = \frac{f_1}{y_1} \Sigma a y^2$$

but $\Sigma a y^2$ is equal to the second moment or moment of inertia, I, of the section about N.A.

$$\therefore M = \frac{f_1}{y_1} I$$

or $$\frac{M}{I} = \frac{f_1}{y_1} = \frac{f}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or $$\frac{M}{I} = \frac{f_2}{y_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

this may also be put in the form

$$M = f_1 \frac{I}{y_1} = f_2 \frac{I}{y_2}$$

$\frac{I}{y_1}$ and $\frac{I}{y_2}$ are called the tension and compression section moduli respectively, and are denoted by Z_1 and Z_2.

$$\therefore M = f_1 Z_1 = f_2 Z_2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is usual to combine equations (1) and (2) into what is called the bending equation, which gives

$$\frac{M}{I} = \frac{E}{R} = \frac{f}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Too much stress cannot be laid on the importance of this equation, and the student is strongly advised to make himself perfectly familiar with it. The units of the various quantities are as follows—

M = bending moment lb. in. (or ton in.).

I = moment of inertia of section about the neutral axis in inch units.

E = Young's modulus in lb./sq. in. (or tons/sq. in.).

R = radius of curvature of neutral line in inches.

f = stress in lb./sq. in. (or tons/sq. in.) due to bending at a distance y from the neutral axis.

68. Application of the Bending Equation to Practical Cases. In practice it is usually found that the bending moment on a beam varies from point to point along the span, and that the bending moment is accompanied by a shearing force. Thus it would appear that the bending equation, which deals with a constant bending moment unaccompanied by a shearing force, is not strictly applicable to such a case.

It will be found, however, that in a great number of practical cases, the bending moment is maximum when the shearing force is zero, and that when the shearing force is maximum the bending moment is almost negligible. Thus the conditions of simple bending are approximated to at the point of maximum bending moment, and hence it seems justifiable to apply the bending equation at this point. The stresses due to the maximum bending moment are usually the most important stresses in a beam, and, therefore, if a beam is designed by the aid of the bending equation to resist the maximum bending moment, its strength will more than suffice at other points along the span where different conditions hold.

EXAMPLE 1.

A rolled joist of I-section has the following dimensions : Flanges 5 in. wide, 0·575 in. thick, web 0·375 in. thick, depth overall 8 in. It is used as a beam freely supported at each end, and covering a clear span of 12 ft. It carries a load of 9 tons uniformly distributed over the span. Calculate the maximum stress produced in the material of the girder due to bending. (Lond. Univ., 1919.)

$$I_{xx} = \frac{5 \times 8^3}{12} - \frac{4{\cdot}625 \times 6{\cdot}85^3}{12}$$

$$= 89{\cdot}33 \text{ in. units}$$

$$M = \frac{WL}{8} = \frac{9 \times 12 \times 12}{8}$$

$$= 162 \text{ ton inches.}$$

$$\frac{M}{I} = \frac{f}{y}$$

$$\therefore f = \frac{M}{I} y$$

$$= \frac{162}{89{\cdot}33} \times 4$$

$$= 7{\cdot}25 \text{ tons/sq. in.}$$

69. **Principal Stresses in Beams.** Shearing stresses exist in beams due to the shearing forces, as well as the direct stresses due to the bending moment. At any point in the beam the direct stress and the shearing stress have a resultant which is the principal stress at the point, and may be found as in par. 22. Curves of principal stress may be obtained by finding the direction of the principal stress at various points in several cross-sections of the beam, and joining up the directions of the principal stresses. Such curves are shown by Fig. 50, drawn on a longitudinal section of a beam.

EXAMPLE 2.

A steel channel 12 in. × $3\frac{1}{2}$ in. has a web $\frac{3}{8}$ in. thick and flanges $\frac{1}{2}$ in. thick. It is used as a beam with the web vertical. At one section of the channel there is a bending moment of 150 in. tons, and a shearing force of 8 tons.

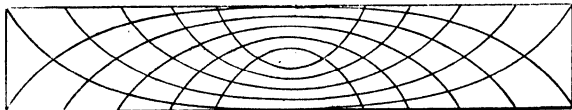



FIG. 50

Find the maximum principal stress at a point 5 in. from the centre of the web. Assume the shear stress to be uniformly distributed over the section of the web. (Lond. Univ., 1916.)

$$I_{xx} \text{ for section} = \frac{3\frac{1}{2} \times 12^3}{12} - \frac{3\frac{1}{8} \times 11^3}{12} \doteqdot 157 \text{ inch units.}$$

Direct stress at point 5 in. from web centre

$$= \frac{My}{I_{xx}} = \frac{150 \times 5}{157} = 4{\cdot}78 \text{ tons/sq. in.}$$

Shear stress on section

$$= \frac{8}{11 \times \frac{3}{8}} = 1{\cdot}94 \text{ tons/sq. in.}$$

$$f_n = \tfrac{1}{2}\{f_x \pm \sqrt{f_x^2 + 4f_s^2}\} \qquad . \qquad . \quad \text{(par. 22)}$$
$$= \tfrac{1}{2}\{4{\cdot}78 \pm \sqrt{22{\cdot}9 + 15{\cdot}1}\}$$
$$= \tfrac{1}{2}\{4{\cdot}78 \pm 6{\cdot}16\}$$

∴ Maximum principal stress = 5·47 tons/sq. in.

70. **Unsymmetrical Bending.** Assumption (5), par. 65, states that for simple bending, the transverse section of the beam must be symmetrical about an axis, passing through the centroid of the section, and parallel to the plane of bending. When

this condition is not fulfilled we may proceed to find the stress at any point in a section, as follows: Let ABC, Fig. 51, be the section of a beam whose centroid is O, and principal axes OX_1 and OY_1. Let OY be the trace of the plane of the applied

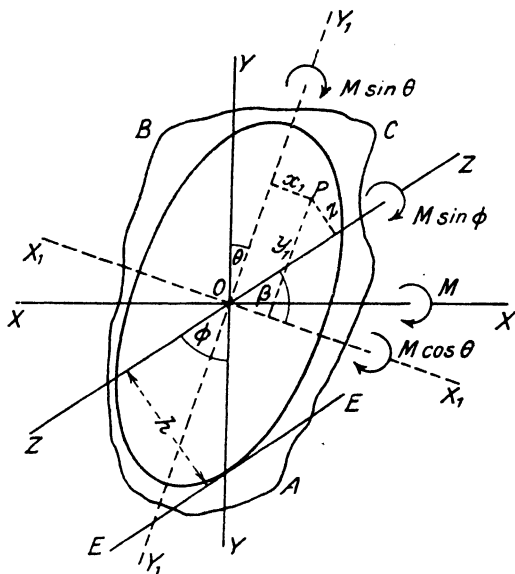



FIG. 51

bending moment M. This bending moment may be resolved into components about the principal axes, namely, $M \cos \theta$ about OX_1 and $M \sin \theta$ about OY_1.

At a point P in the section the stress due to the bending moment $M \cos \theta$ is given by $\dfrac{M \cos \theta \, y_1}{I_{X_1}}$, where y_1 is the distance of P from OX_1 and I_{X_1} is the moment of inertia of the figure about OX_1. Similarly the stress at P due to $M \sin \theta$ is given by $-\dfrac{M \sin \theta \, . \, x_1}{I_{Y_1}}$, taking compression as negative.

$\therefore$ The total stress at P

$$= \frac{M \cos \theta \, . \, y_1}{I_{X_1}} - \frac{M \sin \theta \, . \, x_1}{I_{Y_1}} \quad . \quad . \quad (1)$$

The position of the principal axes may be found as explained in par. 55.

An alternative method depends on the fact that *the neutral axis and the plane of the external bending moment, OY, are conjugate diameters of an ellipse.*

When the point P is on the neutral axis then the stress due to bending is zero, and (1) gives

$$\frac{M\cos\theta \, . \, y_1}{I_{X_1}} = \frac{M\sin\theta \, . \, x_1}{I_{Y_1}} \text{ and } \frac{y_1}{x_1} \text{ becomes } \tan\beta$$

$$\text{or } \frac{y_1}{x_1} = \frac{I_{X_1}}{I_{Y_1}} \tan\theta$$

$$\therefore \tan\beta = \frac{k_x^2}{k_y^2} \tan\theta \qquad (2)$$

$$= \text{the equation of the neutral axis}$$

where k_x and k_y are the semi-principal axes of the momental ellipse. The condition (2) is satisfied by conjugate diameters of the momental ellipse, and since it was obtained by reference to the lines OY and OZ, the trace of the plane of the bending moment M, and the neutral axis respectively, then OY and OZ are conjugate diameters of the momental ellipse. *The plane of bending will be perpendicular to OZ.*

If EE is drawn a tangent to the ellipse and parallel to OZ, then the moment of inertia I_Z about OZ is Ah^2, where h is the perpendicular distance between OZ and EE and A is the area of the section. The bending moment about OZ is $M \sin \phi$ where ϕ is the angle between OZ and OY, and hence the stress at P is given by

$$\frac{M\sin\phi \, . \, z \, .}{I_z} \qquad (3)$$

z being the perpendicular distance from P to OZ.

This method has the advantage that when the neutral axis is drawn, the point in the section at the greatest distance from the neutral axis, and hence the point of maximum stress, is easily seen. On this account, the author recommends the method as being of most general use to students.

EXAMPLE 3.

A 6 in. × 4 in. × ½ in. unequal angle-bar is placed with the long leg vertical and used as a beam supported at each end, the span being 10 ft. What central load can be placed on the angle-bar in order that the maximum stress due to bending may not exceed 7 tons/sq. in.? (Fig. 51A.)

By constructing the momental ellipse we find that the inclination of OY_1 to the longer leg of the angle is $23\frac{1}{2}°$, and that the neutral axis is inclined at 47° to the longer leg, as found by drawing this axis parallel to the tangent where the plane of loading cuts the ellipse, or solving for β in (2).

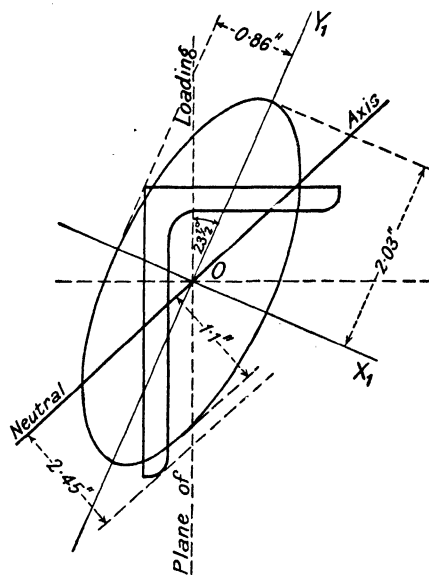



FIG. 51A

Drawing a tangent to the momental ellipse parallel to the neutral axis we find that the radius of gyration about the neutral axis is 1·1 in. The area of the section is 4·75 sq. in.

$$\therefore I_{N.A.} = 4{\cdot}75 \times 1{\cdot}1^2$$
$$= 5{\cdot}75 \text{ inch units.}$$

Also the perpendicular distance z from the neutral axis to the farthest point on the section is 2·45 in., and if M is the applied moment in the plane of loading then $\phi = 47°$.

$$\therefore \quad 7 = \frac{M \sin 47 \times 2{\cdot}45}{5{\cdot}75}$$

$$\therefore M = \frac{7 \times 5 \cdot 75}{2 \cdot 45 \times 0 \cdot 7314}$$

$$= 22 \cdot 4 \text{ ton inches}$$

$$= \frac{W \times 120}{4}$$

$$\therefore W = \frac{4 \times 22 \cdot 4}{120}$$

$$= 0 \cdot 74 \text{ tons}$$

The momental ellipse is drawn to a scale twice as large as that used to construct the angle-bar.

71. **Reinforced Concrete Beams.** Concrete is a material which is strong in compression, but comparatively weak in

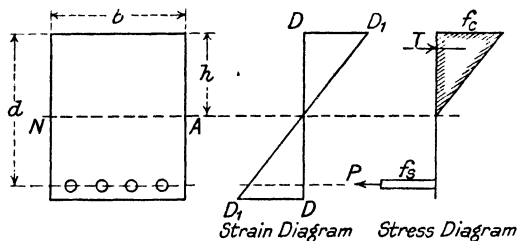



FIG. 52

tension. A beam of such material, therefore, would fail under fairly light loads owing to the tensile stresses due to bending. Additional strength is given to such a beam by embedding in it iron or steel bars in such a position that the bars take the tensile forces. Several theories have been developed to cover the bending of reinforced concrete beams, but the following, called the "*no tension theory,*" is most generally accepted.

The following additional assumptions to the simple bending theory are made—

(1) That there is perfect adhesion between the concrete and the reinforcement.

(2) That all the tensile stress is carried by the reinforcement.

(3) That, in the concrete, the stress is proportional to the strain.

(4) The area of reinforcement is so small that the stress may be assumed constant over it.

A reinforced beam of rectangular section is shown by Fig. 52, with the strain and stress diagrams. The neutral axis of such a beam does not pass through the centroid. Let h be the distance of N.A. from the top of the beam, E_c the value of Young's modulus for concrete, and E_s the value for steel. The ratio $\frac{E_s}{E_c}$ is usually denoted by m.

Since a plane before bending remains a plane after bending, DD will take up the position D_1D_1, and hence the strain is proportional to the distance from the neutral axis.

$$\therefore \frac{\text{strain in concrete at top of beam}}{\text{strain in the steel}} = \frac{h}{d - h} \quad . \quad . \quad (1)$$

also if f_c denotes the stress in the concrete at top of beam,
and if f_s ,, ,, ,, steel, we have

$$\text{strain in concrete at top of beam} = \frac{f_c}{E_c}$$

$$\text{strain in steel} = \frac{f_s}{E_s}$$

$\therefore$ By substitution in (1)

$$\frac{h}{d - h} = \frac{f_c}{E_c} \times \frac{E_s}{f_s} = m\frac{f_c}{f_s} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or
$$hf_s = mf_cd - mf_ch$$

$$\therefore \qquad h = \frac{mf_cd}{f_s + mf_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

An expression which enables us to find the distance h if f_s and f_c are known.

Let A_s be the area of the reinforcement, the total pull P exerted by it is f_sA_s. Since the concrete carries no tension, the stress diagram is triangular and the thrust T exerted by the concrete is given by

$$T = \frac{f_c}{2}\,bh$$

Since the conditions of simple bending are assumed to hold, then

$$P = T$$

and
$$f_sA_s = \frac{f_c}{2}\,bh \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and substituting the value of $\frac{f_c}{f_s}$ as found from (2), we have

$$A_s = \frac{h}{m(d-h)2} \times bh$$

or $bh^2 + 2mA_sh - 2mdA_s = 0$ . . . . . . (5)

a quadratic equation which fixes the value of h, and depends only on known quantities.

The thrust T acts at a distance $\frac{h}{3}$ from the top of the beam, and hence the resisting moment of the beam is given by

$$M = T\left(d - \frac{h}{3}\right)$$

$$= \frac{f_c}{2} bh \left(d - \frac{h}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$M = P\left(d - \frac{h}{3}\right)$$

$$= f_sA_s\left(d - \frac{h}{3}\right) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$M = T(\tfrac{2}{3}h) + P(d - h) \quad . \quad . \quad . \quad . \quad (7)$$

These reinforced beams are often made of T section, and it is very common practice to design the beam so that the neutral axis coincides with the bottom line of the flange of the section. If the neutral axis is lower than this line, we use a similar method of attack to that just used.

EXAMPLE 4

A concrete beam 15 ft. long × 12 in. broad × 15 in. deep, is reinforced by six ¾ in. round steel bars, having their centres 2 in. from the bottom of the beam. Assuming that all the compression is carried by the concrete and all the tension by the steel, determine the uniform loading that may be applied without the stress in the concrete exceeding 600 lb./sq. in., and ascertain the resulting stress in the steel. Take the steel-concrete modular ratio as 15 and the concrete stress strain curve as straight. (Lond. Univ., 1918.)

Area of reinforcement $= \frac{\pi}{4} \times \frac{3}{4} \times \frac{3}{4} \times 6$

$= 2{\cdot}6488$ sq. in.

$$bh^2 + 2mA_sh - 2mdA_s = 0$$

$$12h^2 + 79{\cdot}47h - 1033{\cdot}11 = 0$$

$$\therefore \quad h = \frac{-79{\cdot}47 \pm \sqrt{(79{\cdot}47)^2 + 48 \times 1033{\cdot}11}}{24}$$

or $h = 6{\cdot}55$ in.

$$M = \frac{f_c}{2}\, bh \left(d - \frac{h}{3}\right)$$

$$= \frac{600 \times 12 \times 6{\cdot}55}{2}\left(13 - \frac{6{\cdot}55}{3}\right) = 3600 \times 6{\cdot}55 \times 10{\cdot}8$$

$= 255{,}000$ lb. in.

but $M = \dfrac{WL}{8}$ where W = total load uniformly distributed

$$= \frac{W \times 15 \times 12}{8}$$

$$\therefore \quad W = \frac{255{,}000 \times 8}{15 \times 12} \text{ lb.}$$

$= 5{\cdot}05$ tons

$$f_s A_s = \frac{f_c}{2}\, bh$$

$$\therefore \quad f_s = \frac{600}{2} \times \frac{12 \times 6{\cdot}55}{2{\cdot}649}$$

$= 8900$ lb./sq. in.

72. **Bending Combined with Direct Stress.** Numerous cases occur in which the applied load causes not only stress due to bending, but also exerts a pull or thrust on the section. The resultant stress at any point in the section may be found by calculating the stress due to bending, and superimposing on it the stress due to the direct force ; due regard being given to the sign of each stress.

If M = bending moment
A = area of the section
P = pull or thrust on the section
Z = the section modulus

The resultant stress at any point in the section is given by

$$f = \frac{P}{A} \pm \frac{M}{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the plus sign being used when the bending stress is similar to $\frac{P}{A}$ and the negative sign when it differs from $\frac{P}{A}$. Also, if the section is not symmetrical about a line through the centroid and parallel to the neutral axis, then Z will have the two values Z_1 and Z_2 (par. 67). A diagram of combined stress is shown by (*a*), Fig. 53. The centroid of the section is at G, and, if the

Fig. 53

bending stress acted alone, the neutral axis would pass through G, but, owing to the presence of the direct stress, the point of no stress, and hence the neutral axis, passes through O.

The diagram has been drawn for a case where P is tensile and it may have three possible forms depending on the magnitude of M, Z, P and A.

When $\frac{M}{Z_2}$ is greater than $\frac{P}{A}$ the resultant diagram is given by (*a*)

„ $\frac{M}{Z_2}$ is equal to $\frac{P}{A}$ „ „ „ „ „ (*b*)

„ $\frac{M}{Z_2}$ is less than $\frac{P}{A}$ „ „ „ „ „ (*c*)

The neutral axis, distant y_n from G, is found as follows—

The stress at O due to bending $= \frac{My_n}{I}$ where I is the moment of inertia of the section about an axis through G, or if k is the radius of gyration of the section about this axis then the bending stress at O

$$= \frac{My_n}{Ak^2}$$

At O the bending stress is equal to the direct stress,

$$\text{and } \frac{My_n}{Ak^2} = \frac{P}{A}$$

$$\therefore \quad y_n = \frac{P}{A} \cdot \frac{Ak^2}{M} = \frac{P \, . \, I.}{A \, . \, M}. \qquad (2)$$

$$\text{or } y_n = \frac{Pk^2}{M} \qquad (3)$$

EXAMPLE 5.

A rolled steel joist, 10 in. × 6 in. section, is used as a pillar, and carries an axial load of 25 tons. The maximum moment of inertia of the section is 211·6 in inch units, and the area is 12·36 sq. in. A bracket is bolted to a flange of the pillar and supports a vertical load of 6 tons, which acts in the plane of the major axis of the section, and at a distance of 3 in. from the face of the flange. Calculate the maximum and minimum intensity of stress in a section of the pillar. (A.M.I.Mech.E., 1924.)

The bending moment due to eccentric loading is given by

$$M = 6 \times 8 = 48 \text{ ton inches.}$$

The section modulus $Z = \frac{I}{y}$

$$= \frac{211{\cdot}6}{5} = 42{\cdot}3$$

Then assuming compressive stress as positive, the resultant stress is

$$f = \frac{P}{A} \pm \frac{M}{Z}$$

$$= \frac{31}{12{\cdot}36} \pm \frac{48}{42{\cdot}3}$$

$$= 2{\cdot}508 \pm 1{\cdot}134$$

∴ Maximum stress = 3·642 tons/sq. in. compression.

Minimum " = 1·374 " " "

EXAMPLE 6.

A cylindrical masonry column is 6 ft. in diameter and the maximum wind pressure upon it may be assumed to be equivalent to 20 lb. per square foot of diametrical longitudinal section. If the masonry weighs 140 lb. per cubic foot, to what height can the column be built without causing tension in the cross-section at the base? (Lond. Univ., 1916.)

Let h ft. be the limiting height of the column.

$$\text{Weight of column} = \frac{\pi}{4} \times 36 \times h \times 140$$

$$= 3960h \text{ lb.}$$

$$\text{Total pressure due to wind} = 6 \times h \times 20$$

$$= 120h \text{ lb.}$$

The column may be looked on as a cantilever carrying a direct thrust and a uniformly distributed load, the latter being due to the wind pressure.

Bending moment due to wind pressure

$$= 120h \times \frac{h}{2}$$

$$= 60h^2 \text{ lb. ft.}$$

$$\text{Resultant tensile stress} = \frac{3960h}{28 \cdot 3} - \frac{60h^2 \times 3}{63 \cdot 7} = 0$$

$$\therefore h = \frac{3960 \times 63 \cdot 7}{180 \times 28 \cdot 3}$$

$$= 49 \cdot 5 \text{ ft.}$$

73. **Masonry Structures.** The point of application of the resultant load on the section of a masonry structure is often such that bending and direct stresses are introduced. It is not general practice to assume that masonry can take a tensile load, although anyone who may have observed the destruction of old buildings will often have seen a high wall sway to and fro before collapsing, and an appreciable bend is often to be observed before cracking begins.

The general rules applicable to masonry structures are—

(*a*) That there shall be no tensile stress at any point in the cross-section.

(*b*) That the greatest compression stress is not greater than the safe stress for the masonry.

(*c*) That the shearing force must not be greater than the frictional force between the masonry.

We will only concern ourselves with (*a*), because, when it is fulfilled, the conditions (*b*) and (*c*) are usually fulfilled also.

AB (Fig. 54) represents the cross-section of a masonry structure on which a thrust T, inclined at an angle θ, acts, being distant y from the centroid of the section G. T can be resolved into components $P = T \sin\theta$ perpendicular to AB, and $R = T \cos\theta$ along AB, thus giving a shear at the section.

FIG. 54

If A is the area of the section and I its moment of inertia about G, we have

Direct stress due to $P = \dfrac{P}{A}$

Bending stress due to $P = \dfrac{My_3}{I}$ where y_3 represents any distance from G measured along AB.

But $M = Py$.

$$\therefore \text{ Bending stress} = \frac{Py \times y_3}{I}$$

$$\therefore \text{ Maximum compressive stress} = \frac{P}{A} + \frac{Py}{I} y_2 \quad . \quad (1)$$

and

$$\text{Maximum resultant tensile stress} = \frac{Py}{I} y_1 - \frac{P}{A} \quad . \quad (2)$$

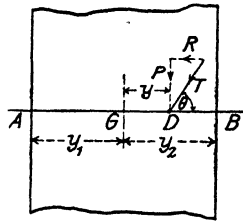
Condition *a* states that this must be zero.

$$\therefore \frac{Py}{I} y_1 = \frac{P}{A}$$

$$\therefore \quad y = \frac{I}{Ay_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This gives the greatest distance that the thrust T may act at from G in order that there may be no tensile stress on the section.

Two well-known sections will now be considered.

(1) Rectangular section where AB is equal to b and the other side equal to a.

From (3)
$$y = \frac{I}{Ay_1}$$
$$= \frac{ab^3}{12} \times \frac{1}{ab \times \frac{b}{2}}$$
$$= \frac{b}{6}$$

i.e. T may be applied at a distance $\frac{b}{6}$ on either side of the centroid without causing tension on the section, or T must lie within the *Middle Third* of the section. This is known as the *Law of the Middle Third.*

(2) Solid circular section where $AB = 2r$.
$$y = \frac{I}{Ay_1}$$
$$= \frac{\pi r^4}{4} \times \frac{1}{\pi r^2 \times r}$$
$$= \frac{r}{4}$$

i.e. T must lie within the *middle quarter* of the section.

Other sections may be worked in a similar manner.

74. **Strain Energy Due to Bending.** From par. 35 the strain energy per unit of volume $= \frac{f^2}{2E}$.

Consider the small length δx of a beam between two transverse sections, and let δA be a small transverse area over which the direct stress is f. If I is the moment of inertia of the whole section about the neutral axis, then

Strain energy in small piece of length δx and area δA
$$= \frac{f^2}{2E} \times \delta x \times \delta A$$
$$= \frac{M^2y^2}{2I^2E}\,\delta x \,.\, \delta A$$

where y is the distance of δA from the neutral axis.

Strain energy in piece of beam between transverse sections

$$= \sum \frac{M^2 y^2}{2I^2 E} \cdot \delta x \cdot \delta A$$

$$= \frac{M^2 \delta x}{2I^2 E} \cdot \Sigma \delta A y^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $\Sigma \delta A y^2 = I$ (par. 51).

$\therefore$ (1) becomes $\frac{M^2 \delta x}{2IE}$

which when δx is reduced indefinitely becomes

$$\frac{M^2 dx}{2IE}$$

$\therefore$ Total strain energy of beam

$$= \frac{1}{2IE} \int_o^L M^2 dx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

L being the span of the beam, and section assumed constant.

(a) For a freely supported beam with a central load W

$$M = \frac{W}{2} x$$

and (2) becomes

$$= \frac{2}{2IE} \int_o^{\frac{L}{2}} \frac{W^2}{4} x^2 \, dx$$

$$= \frac{1}{2IE} \times \frac{W^2 L^3}{48}$$

$$= \frac{W^2 L^3}{96 \, . \, I \, . \, E \, .} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$= \frac{1}{6} \cdot \frac{IL}{y^2 E} \cdot f.^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

(b) For a uniformly loaded cantilever whose load is w per unit run

$$M = \frac{wx^2}{2}$$

$\therefore$ (2) becomes

$$= \frac{1}{2IE}\int_o^L \frac{w^2x^4}{4}\,dx$$

$$= \frac{1}{2IE}\;\frac{w^2L^5}{20}$$

$$= \frac{w^2L^5}{40IE}$$

$$= \frac{W^2L^3}{40IE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{or} = \frac{1}{10}\cdot\frac{IL}{y^2E}\cdot f^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(c) If M is constant (2) becomes

$$= \frac{M^2}{2IE}\int_o^L dx$$

$$= \frac{M^2L}{2IE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

EXAMPLE 7.

A flat ribbon of spring steel, 0·125 in. wide and 0·02 in. thick, is wound round a cylinder 20 in. in diameter. Find the maximum stress in the steel and the energy stored per foot length of the ribbon. The modulus of elasticity E = 30,000,000 lb. per square inch. (Lond. Univ., 1919.)

$$\frac{f}{y} = \frac{E}{R}$$

$$\therefore f = \frac{E}{R}y = \frac{30 \times 10^6 \times 0{\cdot}01}{10}$$

$$= 30{,}000 \text{ lb./sq. in.}$$

Total strain energy

$$= \frac{M^2L}{2IE}$$

$$= \left(\frac{EI}{R}\right)^2 \times \frac{L}{2IE}$$

$$= \frac{EIL}{2R^2}$$

$$= \frac{30 \times 10^6 \times 0{\cdot}125 \times (0{\cdot}02)^3 \times \pi \times 20}{12 \times 2 \times 10 \times 10}$$

$= 0{\cdot}7854$ inch lb.

$\therefore$ Strain energy stored per foot length

$$= \frac{0{\cdot}7854 \times 12}{\pi \times 20}$$

$= 0{\cdot}15$ inch lb.

75. **Shear Stress on Beam Section.** The vertical shearing force on a beam tends to cause sliding on a vertical section, and the shearing stress resulting from this is accompanied at any point in the section by a shearing stress on a horizontal section.

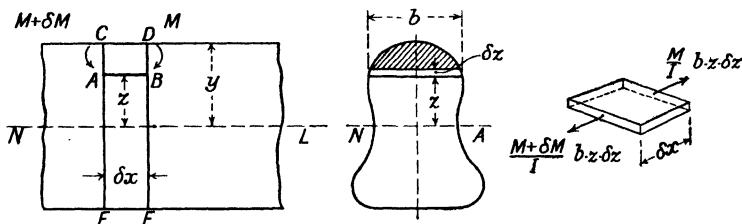



Fig. 55

The two shears cause tensile and compressive forces (par. 18). The compressive force is of importance in a deep built-up girder the web of which is usually thin and thus requires stiffening to prevent failure by buckling. The intensity of shear stress on the section of a beam is not constant from top to bottom of the section, nor is it exactly constant across the width of the section, but, for all practical purposes, we may assume it constant.

The variation in intensity of the vertical shearing force may be found as follows : A beam of uniform section (Fig. 55) has a bending moment M at the section DF, and a bending moment $M + \delta M$ at the section CE ; the two sections being δx apart. Let f be the stress (assumed uniform) at B due to M on a small

area of width b and thickness δz, and f_1 the stress at A on a corresponding area of the cross-section.

$$f = \frac{M}{I} z \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$f_1 = \frac{M + \delta M}{I} z \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Pull on small piece of cross-section due to M

$$= fb\delta z$$

$$= \frac{M}{I} bz\delta z \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Pull on corresponding element of section due to $M + \delta M$

$$= f_1 b\delta z$$

$$= \frac{M + \delta M}{I} bz \, . \, \delta z \quad . \quad . \quad . \quad . \quad (4)$$

The resultant pull on a small slice of length δx, width b, and thickness δz, is found by subtracting (3) from (4)

$$= \frac{\delta M}{I} bz \, . \, \delta z \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the total resultant pull on the piece of length δx and area as shown shaded

$$= \int_z^y \frac{\delta M}{I} bz \, . \, \delta z$$

$$= \frac{\delta M}{I} \int_z^y bz \, . \, \delta z \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This resultant pull is resisted by the shearing force on the longitudinal section at AB whose area is $\delta x \, . \, b$. Let q be the intensity of the stress on this area (assumed uniform), then

$$qb\delta x = \frac{\delta M}{I} \int_z^y bz \, . \, \delta z$$

$$q = \frac{\delta M}{\delta x} \frac{1}{Ib} \int_z^y bz \, . \, \delta z$$

$$= \frac{dM}{dx} \cdot \frac{1}{Ib} \int_z^y bz \, . \, dz$$

But $\frac{dM}{dx} = F$ and $\int_z bz\,.\,dz$ = sum of the moments of the areas of all the small strips comprising the shaded area, about the neutral axis. If A is the shaded area, and $\bar{y}$ the distance of its centroid from N.A. then $\int_z^y bz\,.\,dz = A\bar{y}$

$$\therefore q = \frac{FA\bar{y}}{Ib} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

F being the shearing force at the section.

Since a shear is accompanied by a shear of equal intensity on planes perpendicular to its plane, the intensity of shear on the cross-section at a distance z from the neutral axis is given by q.

(1) Rectangular section. Shear at a distance y from the neutral axis.

$$q = \frac{F}{Ib}\left(b\left(\frac{d}{2} - y\right)\left(y + \frac{\frac{d}{2} - y}{2}\right)\right)$$

$$= \frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right) \quad . \quad . \quad (8)$$

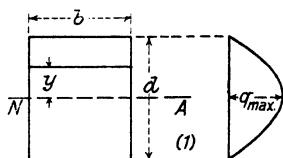



Fig. 56 (1)

which gives a parabola when q and y are plotted (Fig. 56 (1)).

At the neutral axis where $y = 0$, the shear is maximum, and

$$q_{max} = \frac{3}{2}\,\frac{F}{bd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\frac{q_{max}}{q_{average}} = \frac{3}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(2) Similarly for a circular section of diameter d the curve of shear may be shown to be parabolic,

$$\text{and } q_{max} = \frac{4}{3}\,\frac{F}{\frac{\pi d^2}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{q_{max}}{q_{average}} = \frac{4}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(3) The shear diagram for an H section beam is given by Fig. 56 (3), two values of shear being obtained where the web meets the flange, this being due to the two values of b at this point. This diagram, in conjunction with that representing

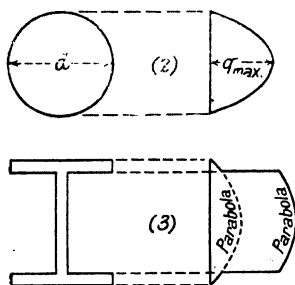



Fig. 56 (2) (3)

the variation of direct stress (Fig. 49), gives us the reason for the common practice amongst engineers, when designing a girder, to assume that the web takes all the shear, and the flanges resist the bending moment.

Example 8.

A beam of I-shaped cross-section is 6 in. deep by 3 in. wide, by 1 in. thickness, and it sustains a shearing force across its section of 5 tons. Draw to scale a diagram showing the distribution of shear stress. (Lond. Univ., 1921.)

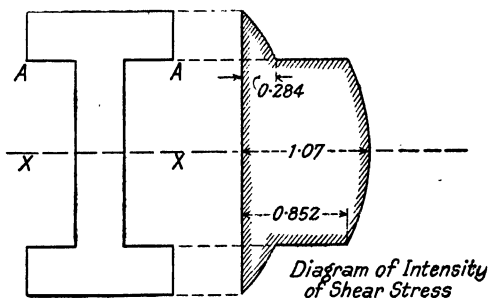



Fig. 56a

$$q = \frac{FA\bar{y}}{Ib} \text{ and } I = \frac{3 \times 6^3}{12} - \frac{2 \times 4^3}{12} \doteqdot 44 \text{ inch units}$$

At top of the section $q = 0$ (Fig. 56a).

Intensity of shear stress in *flanges* at AA is given by

$$q_A = \frac{5 \times 3 \times 2\cdot5}{44 \times 3}$$

$$= 0\cdot284 \text{ tons/sq. in.}$$

Intensity of shear stress in *web* at AA is given by

$$q_A = \frac{5 \times 3 \times 2\cdot5}{44 \times 1}$$

$$= 0\cdot852 \text{ tons/sq. in.}$$

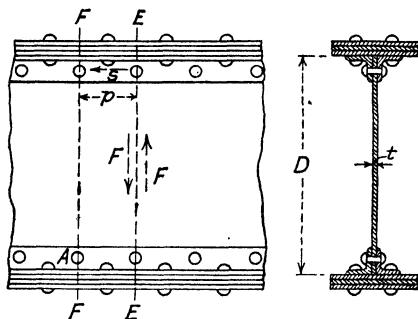



FIG. 57

The distance of the centroid of the area, above or below XX, from XX is found from

$$\{(3 \times 1) + (2 \times 1)\}\bar{y} = (3 \times 1 \times 2\cdot5) + (2 \times 1 \times 1)$$

$$y = \frac{9\cdot5}{5} = 1\cdot9 \text{ in.}$$

$\therefore$ Intensity of shear stress in web at XX

$$= \frac{5 \times 5 \times 1\cdot9}{44 \times 1}$$

$$= 1\cdot078 \text{ tons/sq. in.}$$

76. **Riveting in Built-up Girders.** Large girders, such as those used in travelling cranes and bridges, are usually built up; the web consisting of a single plate connected by angles and rivets to the flanges, which consist of one or more plates (Fig. 57). At any section, such as EE, there is a shearing force F.

The shearing force F is accompanied by an equal shearing

force acting at right angles to F, resisted by the rivets in the web, and from which it is also transmitted to the rivets holding the flanges to the angles. The intensity of shear on the section is not uniform, as has been proved in par. 75, but in the case of deep girders no great error is introduced in assuming the intensity of shear stress to be constant, and that the web takes all the shearing force. A reference to (3), Fig. 56, will show the reason for this assumption.

Let d = the diameter of the rivets

f_s = the intensity of shear stress in the rivets

S = the strength of *one* rivet

Considering the portion of the web between EE and FF, and taking moments of the forces acting thereon about A, we have

$$F \times p = S \times D$$

$$\text{or } p = \frac{SD}{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p is the pitch of the rivets.

Since the rivets are in *double* shear, $S = 2\frac{\pi d^2}{4} f_s$, when the shear strength is less than the crushing strength.

$$\therefore p = \frac{1}{2}\left(\frac{\pi d^2 f_s D}{F}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The rivets attaching the flanges to the angles are in *single* shear and hence the value of S for these rivets will be only half the value for those in the web, consequently if the pitches in the web and flanges are equal, twice as many rivets will be required in the flanges as there are in the web.

If the resistance to crushing is less than the resistance to shear which occurs when the web is very thin, then

$$S = t \times d \times f_c$$

where f_c is the intensity of bearing pressure.

$$\therefore p = \frac{t \times d \times f_c D}{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the value of F is not constant along the girder, the value of p would vary, but for the sake of economy it is usual to make p constant. A common value of d is $\frac{7}{8}$ in. or 1 in.

The vertical and horizontal shear in the web introduce tensile and compressive forces, mutually perpendicular, and inclined at 45° to the direction of the shearing forces.

The compressive force tends to cause the web to buckle, and thus stiffening arrangements are required in deep girders. These consist, usually, of Tee bars, or angle bars, riveted vertically to the web, and spaced a distance apart equal to the depth of the girder.

76A. **Riveted Joint with Eccentric Load.** Fig. 57A represents a riveted joint on which the load P has its line of action at a horizontal distance x from the centroid of the joint. By applying two equal and opposite forces passing through the centroid G, equal in magnitude to P and parallel to its line of action, it will be observed that the joint is subjected to a couple of magnitude Px and a vertical load of magnitude P. The couple causes shearing forces whose directions are shown by the arrows in the diagram, and the vertical load will cause, in each rivet, a vertical shear of magnitude $\frac{P}{n}$, where n is the number of rivets in the joint. The total load on a rivet is the vector sum of the loads acting on it.

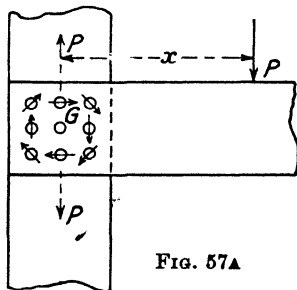



FIG. 57A

If it is assumed that the load on a rivet, due to the couple, is proportional to the relative displacement between the plates, then since this displacement is proportional to the distance of the rivet from G, it follows that the load on a rivet, and hence the stress, is proportional to the distance of the rivet from G.

Let F_s be the shearing force on a rivet at distance r from G, and let f_s be the stress due to F_s, then—

$$f_s = kr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and since $F_s = Af_s$ where A is the cross-sectional area of a rivet

$$\text{then } Px = \Sigma F_s r$$
$$= \Sigma f_s A r$$
$$= \Sigma k A r^2$$
$$\therefore k = \frac{Px}{A\,\Sigma r^2}$$

$$\text{hence } f_s = \frac{Pxr}{A\,\Sigma r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{and } F_s = \frac{Pxr}{\Sigma r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

EXAMPLE 9.

The bracket shown in Fig. 57B is attached to a plate by means of six ¾ in. diameter rivets, and carries a load of 12 tons in the direction shown. Estimate the stress on each rivet.

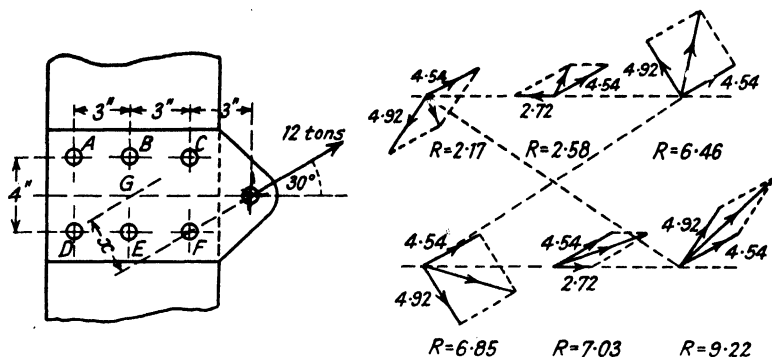



FIG. 57B

$$x = 6 \sin 30° = 3 \text{ in.}$$

$$Px = 36 \text{ ton-in.}$$

$$A\,\Sigma r^2 = \frac{\pi}{4} \times \frac{9}{16} \sum(13 + 4 + 13 + 13 + 4 + 13)$$

$$= 26{\cdot}52$$

$$\text{and } \frac{Px}{A\,\Sigma r^2} = \frac{36}{26{\cdot}52} \doteqdot 1{\cdot}36$$

$\therefore$ for A, C, F and D, $f_s = 1{\cdot}36 \times 3{\cdot}61 = 4{\cdot}92$ tons/sq. in.
and for B and E $\qquad f_s = 1{\cdot}36 \times 2 \qquad = 2{\cdot}72$ tons/sq. in.

The shear stress on each rivet due to P acting through G is given by

$$\frac{12}{6 \times \frac{\pi}{4} \times \frac{9}{16}} = 4{\cdot}54 \text{ tons/sq. in.}$$

The resultant stress on each rivet is found, as shown in Fig. 57B, in both magnitude and direction.

These are as follows—

Rivet	*A*	*B*	*C*	*D*	*E*	*F*
Resultant shear stress—tons/sq. in. . .	2·17	2·58	6·46	6·85	7·03	9·22

EXAMPLES VII

1. A floor has to carry a load of 3 cwt. per square foot. The floor joists are 12 in. deep by $4\frac{1}{2}$ in. thick, and have a span of 14 ft. Determine the distance apart from centre to centre at which these joists must be spaced, if the maximum stress is not to exceed 1,000 lb./sq. in. (Lond. Univ., 1916.)
Ans., 13·1 in.

2. An overhung steel crank pin journal is so designed that the pressure on the journal is limited to 600 lb./sq. in. of projected area. The total load on the journal is 60,000 lb., and the maximum bending stress is limited to 10,000 lb./sq. in. Find dimensions for the diameter and length of the journal. (Lond. Univ., 1918.) *Ans.*, $7\frac{1}{2}$ in. and $13\frac{3}{4}$ in.

3. The moment of inertia of a beam section 20 in. deep, is 1,670 in. units. Find the longest span over which a beam of this section, when simply supported, could carry a uniformly distributed load of 1·5 tons per foot run. The maximum flange stress in the material is not to exceed 7 tons/sq. in. (A.M.I.Mech.E., 1918.) *Ans.*, 22·8 ft.

4. A plank of timber 8 ft. long and 2 in. thick has a width varying uniformly from 18 in. at one end to 10 in. at the other end. It is supported at its ends with its length and width horizontal. If the density of the timber is 55 lb./cub. ft., find the maximum bending moment in the plank due to its own weight and the maximum longitudinal stress produced in the wood by this bending moment. (Lond. Univ., 1922.)
Ans., 1,030 lb. in. (at 50 in. from small end), 109 lb./sq. in.

5. A flitched timber beam consists of two timber joists each 4 in. wide by 12 in. deep, with a steel plate $\frac{3}{4}$ in. thick and 8 in. deep, placed symmetrically between them and firmly fixed in place. If the span is 20 ft., and the ends are simply supported, calculate the maximum uniformly distributed load the beam can carry if the stress intensity in the timber is not to exceed 1,000 lb./sq. in. What will then be the maximum stress in the steel ? E for steel = 30,000,000 lb./sq. in., E for timber, 1,500,000 lb./sq. in. (Lond. Univ., 1914.)
Ans., 489 lb./ft. run, 13,333 lb./sq. in.

6. A simple bridge is formed of telegraph poles, laid side by side, with all the butt ends on the one abutment. If Z is the modulus of the section at the butt end, and Z_x the modulus of the section x inches from the butt end, $Z_x = Z - 0{\cdot}3x$. Span = 30 ft. Diameter of poles at butt ends 12 in. Find the position of the most highly stressed section when the bridge is uniformly loaded throughout its length. (Lond. Univ., 1915.) *Ans.*, $x = 18{\cdot}7$ ft.

7. A 7 in. × $3\frac{1}{2}$ in. × $\frac{1}{2}$ in. unequal angle bar is placed with the longer leg vertical and used as a beam simply supported at each end. Find what

uniformly distributed load can be spread over the span of 12 ft. in order that the maximum stress due to bending may not exceed 7 tons/sq. in.
Ans., 312 lb./ft. run.

8. A reinforced concrete slab, $7\frac{1}{2}$ in. thick, has an effective span of 10 ft. The reinforcement consists of $\frac{1}{2}$ in. diameter bars at 6 in. centres placed $1\frac{1}{2}$ in. above the bottom of the slab. Determine what uniform load per square foot the slab will carry in addition to its own weight if the allowable maximum stresses are 18,000 lb./sq. in. for steel and 650 lb./sq. in. for concrete, and if the ratio of the modulus of elasticity of steel to that of concrete is 12. Weight of slab 150 lb./cub. ft. (Lond. Univ., 1923.) *Ans.*, 160 lb.

9. A reinforced concrete T-beam has a flange 60 in. wide and 4 in. deep. The reinforcement is placed in the rib 15 in. from the upper edge of the flange. The beam is designed so that the neutral axis coincides with the lower edge of the flange. The limits of stress are for steel 16,000 lb./sq. in., and for concrete 600 lb./sq. in. The ratio $\frac{E \text{ steel}}{E \text{ concrete}}$ is 15. Calculate: (*a*) the area of the reinforcement, (*b*) the moment of resistance of the beam, (*c*) the actual maximum stress in the steel and in the concrete. (Lond. Univ., 1916.)
Ans., 2·9 sq. in., 636,000 lb. in., $f_s = 16{,}000$ lb./sq. in., $f_c = 388$ lb./sq. in.

10. A short circular column, external diameter 10 in., thickness of metal 1 in., carries a load of 50 tons, the line of action of which coincides with the axis of the column. It also carries a second load of 15 tons, whose line of action is 6 in. from the centre. Find the maximum and minimum stresses in the column, and show by a diagram how the stresses vary across the section. (Lond. Univ., 1918.)
Ans., 3·84 tons/sq. in. comp., 0·744 tons/sq. in. compression.

11. A steel plate chimney, 4 ft. in diameter and 80 ft. high has a cylindrical bottom ring $\frac{3}{8}$ in. thick. The weight of the structure may be considered as that of a cylinder of this thickness, and the internal stiffening devices increase the second moment of area of the section 60 per cent. Find the greatest stress in the plates when the chimney is subjected to a uniform wind pressure of 40 lb./sq. ft. of projected area. Take steel as weighing 480 lb./cub. ft. (Lond. Univ., 1913.) *Ans.*, 6,100 lb./sq. in.

12. A short stanchion is made of I-section, 6 in. wide and 8 in. deep. Find the greatest and least intensity of compressive stress on a section if the load of 50 tons has its centre in the centre line of the section which bisects both flanges, but 1·6 in. from the centre line which is parallel to the outer edges of the flanges. The moment of inertia of the cross-section about a central axis parallel to the outer edges of the flanges is 110·5 (inches)^4 and the area of the cross-section is 10·29 sq. in. (A.M.I.Mech.E., 1925.)
Ans., 7·755 and 1·965 tons/sq. in. compression.

13. A steel plate 4 in. wide and $\frac{3}{4}$ in. thick is pin-jointed at the ends and subjected to a pull of 5 tons. The plate is horizontal with the 4-in. width vertical and the axis of the load is $\frac{1}{4}$ in. above the axis of the plate. The distance between the centres of the pins is 7 ft. What load must be hung at the centre of the span so that the distribution of longitudinal stress at this section may be uniform? Draw a diagram showing the distribution of longitudinal stress under these circumstances at a vertical section 2 ft. from a pin. (Lond. Univ., 1922.) *Ans.*, 267 lb.

14. The section of a brick pier is a rectangle and hollow, the dimensions of the external and internal rectangles being 54 in. by 45 in., and 36 in. by 27 in. respectively. Find the maximum distance from the centre of the point through which the line of action of the resultant thrust may pass in order that there may be no tension in the section. (Lond. Univ., 1912.) *Ans.*, 12·3 in.

15. The mean intensity of bearing pressure on the projected area of an overhung crank pin is 800 lb./sq. in. The bending stress in the pin is 8,500 lb./sq. in. Find the diameter and length of the pin, also the maximum intensity of shear stress. The total load on the pin is 90,000 lb. (Lond. Univ., 1915.) *Ans.*, $8\frac{7}{8}$ in., $12\frac{7}{8}$ in., 1,955 lb./sq. in.

16. Show that the distribution of shearing stress across the vertical section of a loaded beam of rectangular cross-section is parabolic. Find the maximum shearing stress produced by a load of 20 tons in the vertical section of a hollow beam of square section if the outside width is 5 in. and the thickness of the material 1 in. (Lond. Univ., 1922.) *Ans.*, 2·7 tons/sq. in.

17. The vertical cross-section of a loaded horizontal beam is an isosceles triangle, base 4 in., height 6 in. Draw a diagram, showing the distribution of shearing stress across the section where the shear load is 6 tons. (Lond. Univ., 1923.)

18. Show that the elastic energy of any rod, stressed in a particular manner, is proportional to the square of the stress at any one place. A circular rod, diameter d, length l, is loaded with a weight W, (a) in direct tension, (b) centrally on a simply supported span. Compare the strain energies in the two conditions. (Lond. Univ., 1921.) *Ans.*, $\frac{3d^2}{l^2}$

19. Find an expression for the resilience of a bar of uniform circular section when subjected to a uniform bending moment. A steel bar of 2 in. diameter is supported at the ends and carries a load at the centre of the span which is 30 in. If the maximum stress induced is 15,000 lb./sq. in., find the number of foot-pounds of elastic energy stored up in the bar. $E = 30,000,000$ lb./sq. in. (Lond. Univ., 1918.) *Ans.*, 2·45 ft. lb.

20. Calculate the cross-sectional dimensions of the strongest rectangular beam that can be cut out of a cylindrical log of wood whose diameter is 10 in. (A.M.I.C.E., 1925.) *Ans.*, 5·775 in $\times$ 8·15 in.

21. A hollow circular tie-bar, whose outside diameter is 3 in. and internal diameter is 2 in., is pulled in such a way that the axis of the pull is parallel to but not concentric with the axis of the bar. Calculate the amount of the eccentricity which can be allowed if the maximum stress in the material is not to exceed the mean stress by more than 25 per cent. (A.M.I.C.E., 1926.) *Ans.*, 0·134 in.

22. The web plate of a girder is 60 in. high and $\frac{3}{8}$ in. thick; and the flanges each consist of one 14 in. $\times$ $\frac{1}{2}$ in. plate and are joined to the web by $6 \times 6 \times \frac{1}{2}$ in. angles and $\frac{7}{8}$ in. rivets. Find a suitable pitch for the rivets if the total shearing force is 62·6 tons and the allowable load per rivet is 3·4 tons. *Ans.*, $3\frac{1}{4}$ in.

23. A horizontal chain passes over a pulley carried by a bracket attached to a stanchion by six $\frac{7}{8}$ in. rivets as shown in Fig. 57c. If the rivets are in single shear and the shear stress is limited to $5\frac{1}{2}$ tons/sq. in., find the load W which can be carried from the free end of the chain. (Lond. Univ., 1933.) *Ans.*, 1·6 tons.

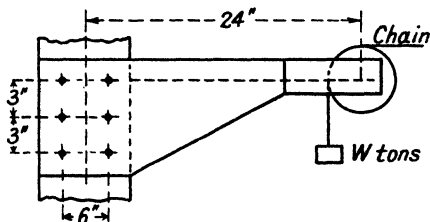



FIG. 57c

CHAPTER VIII

SLOPE AND DEFLECTION

77. IN Chapter VII the *strength* of beams has been investigated. In this chapter the beam problem will be approached from an equally important direction, namely, with regard to *stiffness*. The total deflection of a beam is due, to a very large extent, to the deflection caused by bending, and, to a very much smaller extent, to the deflection caused by shear. In practice it is usual to put a limit on the allowable deflection, hence it is important that we should be able to calculate the deflection of a beam of given section, under certain conditions of loading, before making practical use of the beam, since, for given conditions of span and load, it would be possible to adopt a section which would be quite strong enough, but the deflection would be abnormal.

In calculating the deflection due to bending, two methods will be used : one depending on a knowledge of the area of the bending moment diagram, the other being purely mathematical. The former method, after careful study, will be found to simplify many problems, which, by the mathematical method, are far from simple. It will also be found to lend itself to the easy solution of "fixed beams," and "continuous beams," or those resting on more than two supports. It should be borne in mind that in actual beam problems the slope at any point is exceedingly small, and hence the radian measure of the angle of slope is approximately equal to the tangent of the angle of slope.

78. **Deflections from a Knowledge of the M Diagram.** In Fig. 58 a portion of a beam of length BD has a bending moment diagram of area A represented by BCD. The distance of the centroid G of the diagram from any chosen vertical line HH is $\bar{x}$. An exaggerated view of the deflected beam is shown below the bending moment diagram.

Consider a small piece of the beam of length δx over which the bending moment may be assumed to be constant, and equal to M. The change of slope over the small piece δx is given by $\delta\theta$, where $\delta\theta$ is the angle included between tangents drawn at each extremity of δx. Let R be the radius of curvature of the

small length δx when deflected, then since $\delta\theta$ is very small we have that,

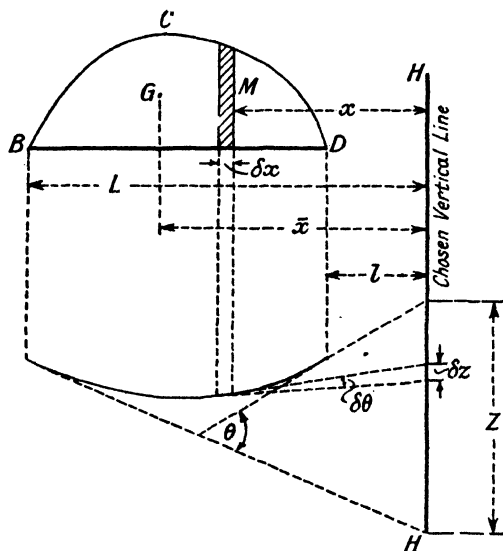
$$\delta\theta = \frac{\delta x}{R}$$

$$\text{but } \frac{M}{EI} = \frac{1}{R}$$

hence
$$\delta\theta = \frac{1}{EI} M\delta x$$

FIG. 58

$$\therefore \qquad \theta = \int_l^L \frac{1}{EI} M \,.\, dx \,.$$

$$= \frac{1}{EI} \int_l^L M \,.\, dx$$

when the section of the beam is constant

$$= \frac{1}{E} \int_l^L \frac{M}{I} \, dx$$

when the section of the beam varies,

but $\int_l^L M \,.\, dx$. is the area A of the bending moment diagram BCD over the whole piece of length BD.

$$\therefore \qquad \theta = \frac{A}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

θ being the change of slope over the whole piece of the beam of length BD. Thus we have the important relationship between the bending moment diagram, over *any portion* of a loaded beam, and the change of slope over the *same portion*, *that the change of slope is equal to the area of the bending moment diagram divided by EI.*

$$\text{Also} \qquad \delta Z \doteqdot x\delta\theta$$

$$= x \frac{M}{EI} \delta x$$

$$\therefore \qquad Z = \frac{1}{EI}\int_l^L Mxdx \text{ when } EI \text{ is constant}$$

$$\text{but} \int_l^L Mxdx = A\bar{x}$$

$$\therefore \qquad Z = \frac{A\bar{x}}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the intercept, on any chosen line, between the tangents drawn to the ends of any portion of a loaded beam, is equal to the product of the area of the bending moment diagram, over that portion of the beam, and the distance of the centre of gravity of this diagram from the chosen line, divided by EI.

If the vertical line is properly chosen, then Z will represent the deflection of the beam. In the case where the bending moment diagram is not all of the same sign, care must be taken in finding the value of $A\bar{x}$. Examples of this will be found later.

Since the area of the bending moment diagram is proportional to the product of W and L^2, where W is the total load on the beam, and x is proportional to L, then the deflection will be given by

$$Z = k\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In Fig. 59 ACB is the deflected form of a loaded beam, greatly

exaggerated. θ_A and θ_B represent the slope at A and B respectively; the tangents to the curve at these points being AD and BD. At C, the point of maximum deflection, draw FE tangent to the curve, and intersecting BD in G, and a perpendicular to AB at B, in E. At A draw a perpendicular to AB, to cut the tangent at C in F, and meet BD produced in K.

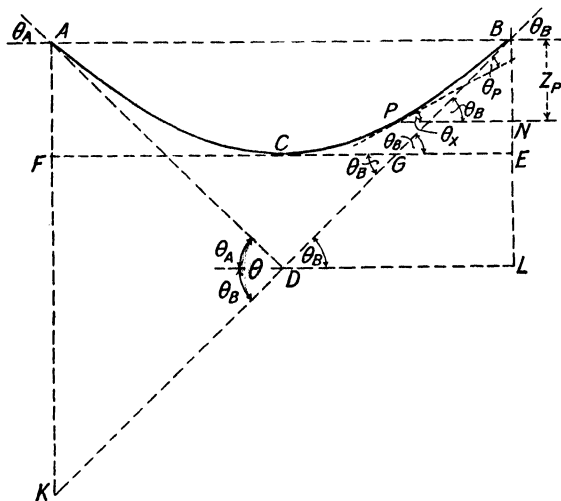



FIG. 59

Let A = area of bending moment diagram above AB

$\bar{x}$ = distance of the centroid of the bending moment diagram from the end A of the beam

then $Z = AK$

$= BE + FK$, and since θ_B is a small angle

$= GE \,.\, \theta_B + FG \,.\, \theta_B = \theta_B(GE + FG) = \theta_B.AB$

$= l\theta_B$ where $AB = l$

but $Z = \dfrac{Ax}{EI}$ from (2).

$\therefore\ l\theta_B = \dfrac{Ax}{EI}$

$$\text{and } \theta_B = \frac{A\bar{x}}{EIl} \qquad \qquad (4)$$

similarly $\theta_A = \dfrac{A\,(l - \bar{x})}{EIl}$ . . . . (5)

also $\theta = \theta_A + \theta_B$ . . . . (6)

and θ_x, the slope of the beam at any point P

$= \theta_B - \theta_P$ . . . . (7)

where θ_P is the change of slope over PB.

$$\therefore Z_P = \frac{A_P \bar{x}_P}{EI} + PN\,\theta_x \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where A_P is the area of the bending moment diagram above PN.

$\bar{x}_P$ = the distance of the centroid, of the bending moment diagram above PN, from BL

PN = the horizontal distance of P from BL,

and substituting in (8) the value of θ_x from (7) we have,

$$Z_P = \frac{A_P \bar{x}_P}{EI} + PN\,(\theta_B - \theta_P)$$

$$= \frac{A_P \bar{x}_P}{EI} + PN\,\frac{A\bar{x}}{EIl} - \frac{A_P \,.\, PN}{EI} \quad . \quad . \quad . \quad (9)$$

This equation enables us to find the deflection at *any* point along the span, but care must be taken when the slope of the beam changes sign.

EXAMPLE 1.

A rectangular beam of wood, 12 ft. long, 3 in. wide, and 4 in. deep, is supported at its ends, and is loaded with loads of 300 lb. and 500 lb., placed at points 5 ft. and 8 ft. respectively from one end. Calculate the maximum bending moment produced by the loading and find the deflection produced by bending at the mid-point of the beam. (Lond. Univ., 1923.)

$$R_B = \frac{(300 \times 5) + (500 \times 8)}{12} = \frac{5500}{12} \text{ lb.} \quad \text{(Fig. 59A)}$$

$$M_C = \frac{5500}{12} \times 4 \times 12 = 22{,}000 \text{ lb. in.}$$

$$M_D = \left(\frac{5500 \times 7 \times 12}{12}\right) - (500 \times 3 \times 12)$$

$$= 20{,}500 \text{ lb. in.}$$

Thus the maximum bending moment is 22,000 lb. in.

M increases from D to C at a uniform rate of 500 lb. in. per ft. run.

$\therefore M$ at mid point $= 20{,}500 + 500$
$= 21{,}000$ lb. in.

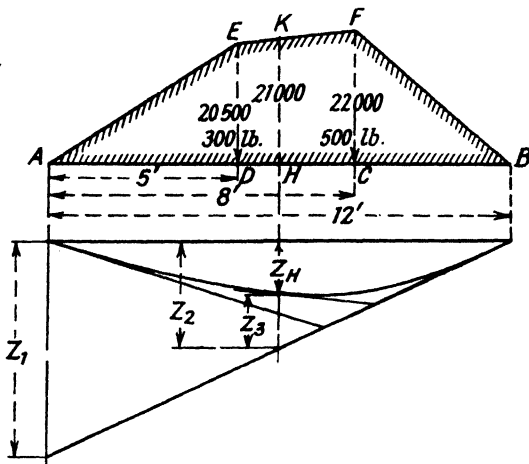



FIG. 59A

$$I = \frac{bd^3}{12} = \frac{3 \times 4^3}{12} = 16 \text{ inch units.}$$

$$Z_1 = \frac{144}{EI}\left\{\frac{20{,}500 \times 5 \times 10}{2 \times 3} + \frac{20{,}500 \times 3 \times 13}{2} + \frac{1500 \times 3 \times 7}{2} + \frac{22{,}000 \times 4 \times 28}{2 \times 3}\right\}$$

$$= 4{\cdot}48 \text{ in.}$$

$$Z_2 = \tfrac{1}{2}Z_1 = 2{\cdot}24 \text{ in.}$$

$$Z_3 = \frac{144}{EI}\left\{(21{,}000 \times 2 \times 1) + \frac{1000 \times 2 \times 4}{2 \times 3} + \frac{22{,}000 \times 4 \times 10}{2 \times 3}\right\}$$

$$= 0{\cdot}852 \text{ in.}$$

$$Z_H = Z_2 - Z_3 = 2{\cdot}24 - 0{\cdot}852$$

$$= 1{\cdot}388 \text{ in. for a value of } E = 2 \times 10^6 \text{ lb./sq. in.}$$

79. Examples Illustrating the Application of $Z = \frac{Ax}{EI}$

(*a*) Horizontal beam freely supported at each end, with an isolated load W at mid-span.

The M diagram and the deflected form of the beam are given by Fig. 60. If the vertical line is chosen so that it passes

Fig. 60

through a support, then, considering *one-half* of the span, the intercept between the tangents is Z.

Also $$A = \frac{WL^2}{16} \text{ and } \bar{x} = \frac{2}{3}\frac{L}{2} = \frac{L}{3}$$

$$\therefore Z = \frac{WL^2}{16} \times \frac{L}{3} \times \frac{1}{EI}$$

$$= \frac{1}{48} \cdot \frac{WL^3}{EI} \qquad \qquad (1)$$

At a point distant x from the centre of the span, the deflection is given by

$$Z_x = Z_1 + Z_2$$

$$= Z_1 + \left(\frac{L}{2} - x\right)\theta_2 \text{ since } \theta_2 \text{ is very small.}$$

$$Z_1 = \frac{W}{2}\frac{\left(\frac{L}{2} - x\right)^2}{2} \times \frac{2}{3}\left(\frac{L}{2} - x\right)\frac{1}{EI} = \frac{W}{6EI}\left(\frac{L}{2} - x\right)^3$$

$$\theta_2 = \frac{W}{4}(L - x)\,x\,\frac{1}{EI}$$

$$\therefore Z_2 = \frac{W}{4}(L - x)x\frac{1}{EI} \times \left(\frac{L}{2} - x\right) = \frac{W}{4EI}(Lx - x^2)\left(\frac{L}{2} - x\right)$$

$$\therefore Z_x = Z_1 + Z_2 = \frac{W}{EI}\left\{\frac{1}{6}\left(\frac{L}{2} - x\right)^3 + \frac{1}{4}(Lx - x^2)\left(\frac{L}{2} - x\right)\right\}$$

$$= \frac{W}{EI}\left(\frac{L^3}{48} - \frac{Lx^2}{8} + \frac{x^3}{12}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

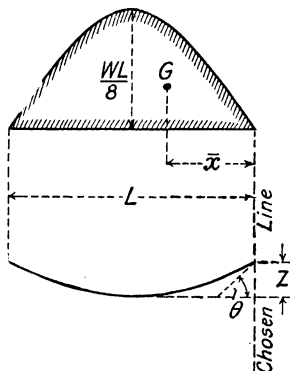



FIG. 61

At mid-span where $x = 0$, equation (2) reduces to the value obtained in equation (1).

(*b*) Horizontal beam freely supported at each end, with a load W uniformly distributed over the span.

Referring to Fig. 61, and again considering *one-half* of the span,

$$A = \frac{2}{3}\frac{WL}{8} \times \frac{L}{2} = \frac{WL^2}{24}, \text{ and } \bar{x} = \left(\frac{5}{8}\cdot\frac{L}{2}\right) = \frac{5}{16}L$$

$$\therefore Z = \frac{5}{384}\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*c*) Cantilever with isolated load not at the free end.

If the load is at a distance l from the fixed end (Fig. 62) then $A = \frac{Wl^2}{2}$, and taking the chosen line through the line of action of W, then the deflection at $W = Z_w$, and

$$x = \tfrac{2}{3} l$$

$$\therefore Z_w = \frac{Wl^2}{2} \times \frac{2}{3} l \times \frac{1}{EI}$$

$$= \frac{1}{3} \frac{Wl^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

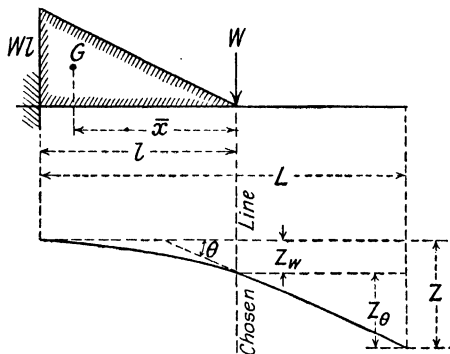



FIG. 62

The deflection at the free end is given by $Z = Z_w + Z_\theta$

or $Z = Z_w + (L - l)\,\theta$.

$$\theta = \frac{Wl^2}{2EI}$$

$$\therefore Z = \frac{W}{EI}\left(\frac{l^3}{3} + \frac{l^2}{2}(L - l)\right)$$

$$= \frac{W}{EI}\left(\frac{Ll^2}{2} - \frac{l^3}{6}\right) \quad . \quad . \quad . \quad . \quad (5)$$

(*d*) Cantilevèr with isolated load at free end. In (4) let $l = L$,

then $$Z = \frac{1}{3} \frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(*e*) Cantilever with uniform load over the span.

The area of the bending moment diagram (Fig. 63) is given by

$$A = \frac{1}{3}\frac{WL}{2} \times L = \frac{WL^2}{6} \text{ and } \bar{x} = \frac{3}{4}L$$

$$\therefore Z = \frac{WL^2}{6} \times \frac{3}{4}L \times \frac{1}{EI}$$

$$= \frac{1}{8}\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

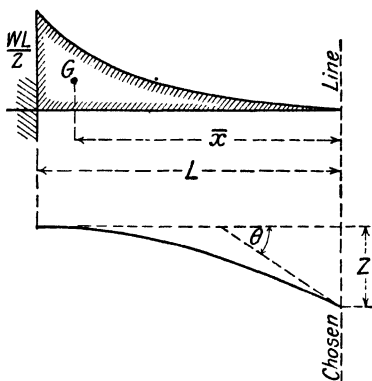
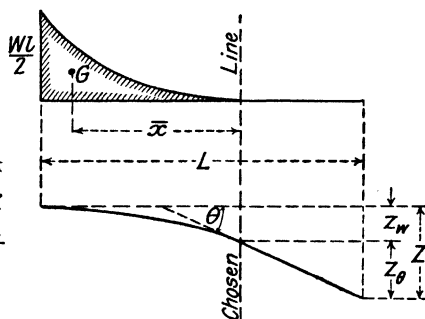



FIG. 63 FIG. 64

(*f*) Cantilever with uniform load from fixed end for a distance l of the span.

The total deflection $Z = Z_\theta + Z_W$ (Fig. 64). Taking the chosen line at the end of the uniform load, we have

$$Z_W = \frac{1}{8}\frac{Wl^3}{EI} \text{ and } \theta = \frac{Wl^2}{6EI}$$

$$\text{and } Z_\theta = (L - l)\theta = \frac{Wl^2}{6EI}(L - l)$$

$$\therefore Z = \frac{W}{EI}\left(\frac{Ll^2}{6} - \frac{l^3}{6} + \frac{l^3}{8}\right)$$

$$= \frac{W}{EI}\left(\frac{Ll^2}{6} - \frac{l^3}{24}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

EXAMPLE 2.

A cantilever of uniform section is loaded with 4 tons on the extreme end and 6 tons uniformly distributed. Calculate the deflection at the free end. Length 14 ft., depth of section 18 in., maximum stress due to bending 5 tons/sq. in., Young's modulus of elasticity 13,000 tons/sq. in. (Lond. Univ., 1915.)

Maximum bending moment

$$= (4 \times 14 \times 12) + (6 \times 7 \times 12)$$
$$= 1176 \text{ ton in.}$$

$$\frac{M}{I} = \frac{f}{y}$$

$$I = \frac{My}{f} = \frac{1176 \times 9}{5} = 2117 \text{ inch units}$$

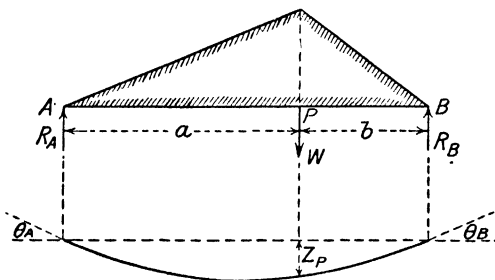



FIG. 65

Total deflection $= \frac{1}{3}\frac{W_1L^3}{EI} + \frac{1}{8}\frac{W_2L^3}{EI}$

$$= \frac{L^3}{EI}\left\{\frac{W_1}{3} + \frac{W_2}{8}\right\}$$

$$= \frac{L^3}{EI}\left\{\frac{4}{3} + \frac{6}{8}\right\}$$

$$= \frac{(14 \times 12)^3 \times 25}{13{,}000 \times 2117 \times 12}$$

$$= 0{\cdot}36 \text{ in.}$$

(*g*) Horizontal beam freely supported at each end with an isolated load not at mid-span.

We have from Fig. 65 $R_A = \frac{Wb}{a+b}$ and $R_B = \frac{Wa}{a+b}$

From (4), par. 78 $\theta_B = \dfrac{A\bar{x}}{EIl}$

$$= \frac{1}{EI\,(a+b)}\left\{\frac{Wb}{a+b}\left(\frac{a^2}{2}\times\frac{2}{3}a\right)+\frac{Wa}{a+b}\cdot\frac{b^2}{2}\left(a+\frac{b}{3}\right)\right\}$$

$$= \frac{Wab\,(2\,a+b)}{6EI\,(a+b)} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also θ_A will be $= \dfrac{Wab\,(2\,b+a)}{6EI\,(a+b)}$ . . . . . (2)

From (7), considering the portion A to P, we have slope at X distant x from A,

$$\theta_x = \theta_A - \frac{1}{EI}\text{ area above } AX$$

$$= \frac{Wab\,(2b+a)}{6EI\,(a+b)} - \frac{1}{EI}\,\frac{Wb}{a+b}\times\frac{x^2}{2}$$

$$= \frac{Wb}{EI\,(a+b)}\left\{\frac{2ab+a^2}{6}-\frac{x^2}{2}\right\} \qquad . \quad . \quad . \quad (3)$$

At the point of maximum deflection $\theta_x = 0$

$$\therefore x^2 = \frac{a^2+2ab}{3} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The deflection at X is given by Z_X

$$= \frac{Wb}{EI(a+b)}\left\{x\times\frac{x}{2}\times\frac{2}{3}x\right\}+\frac{Wb}{EI\,(a+b)}\left(\frac{2abx+a^2x}{6}-\frac{x^3}{2}\right)$$

$$= \frac{Wbx}{EI(a+b)}\left\{\frac{x^2}{3}+\frac{2ab+a^2}{6}-\frac{x^2}{2}\right\}$$

$$= \frac{Wbx}{EI(a+b)}\left\{\frac{2ab+a^2-x^2}{6}\right\} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The deflection under the load is given by substituting $x = a$ in (4),

$$\therefore Z_P = \frac{Wba}{EI(a+b)}\left\{\frac{2ab+a^2-a^2}{6}\right\}$$

$$= \frac{Wa^2b^2}{3(a+b)EI} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The maximum deflection is obtained by substituting the value of x^2 from (4) in (5),

$$Z_{max} = \frac{Wb\sqrt{\frac{a^2 + 2ab}{3}}}{EI(a+b)}\left(\frac{2ab + a^2}{6} - \frac{a^2}{18} - \frac{2ab}{18}\right)$$

$$= \frac{Wb\sqrt{\frac{a^2 + 2ab}{3}}}{EI(a+b)}\left(\frac{a^2 + 2ab}{9}\right)$$

$$= \frac{Wb(a^2 + 2ab)^{\frac{3}{2}}}{9\sqrt{3}EI(a+b)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

EXAMPLE 3.

Compare the magnitudes of the slopes which occur at each end of a freely supported beam placed across a span of L ft., when a weight of W lb. is concentrated at a spot one-third of the span from one end. Assume the beam to be horizontal when W is removed. (Lond. Univ., 1922.)

Referring to Fig. 65. $R_B = \frac{W \times 2L}{3 \times L} = \frac{2}{3}W$ and $R_A = \frac{1}{3}W$

$$M_P = R_B \times b$$

$$= \frac{2}{3}W \times \frac{1}{3}L$$

$$= \frac{2}{9}WL \text{ lb. ft.}$$

$$\theta_B = \frac{1}{EIL}\left(\frac{2}{9}\frac{WL}{2} \times \frac{2L}{3} \times \frac{4}{9}L + \frac{2}{9}\frac{WL}{2} \times \frac{1}{3}L \times \frac{7}{9}L\right)$$

$$= \frac{WL^3}{EIL}\left(\frac{15}{243}\right) = \frac{15}{243}\frac{WL^2}{EI}$$

$$\theta_A = \frac{1}{EIL}\left(\frac{2}{9}\frac{WL}{2} \times \frac{1}{3}L \times \frac{2}{9}L + \frac{2}{9}\frac{WL}{2} \times \frac{2}{3}L \times \frac{5}{9}L\right)$$

$$= \frac{WL^3}{EIL}\left(\frac{12}{243}\right) = \frac{12}{243}\frac{WL^2}{EI}$$

$$\therefore \frac{\theta_A}{\theta_B} = \frac{12}{15} = \frac{4}{5}$$

(*h*) Beam subjected to a uniform bending moment.

Let the magnitude of the bending moment be M, and the span L. The beam will bend to an arc of a circle, and the greatest deflection will occur at mid-span. Considering one-half of the span, and taking the chosen vertical line through one end of the beam, we have

$$A = M\frac{L}{2} \quad \text{and } \bar{x} = \frac{L}{4}$$

$$\therefore Z = \frac{ML^2}{8EI}$$

(i) Beam with overhanging end to which a couple is applied.

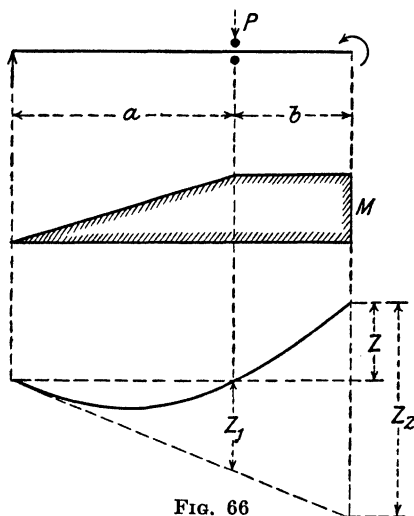



FIG. 66

The M diagram, and the deflected form of the beam, are shown by Fig. 66.

The deflection at the free end is given by

$$Z = Z_2 - Z_1\frac{a+b}{a}$$

and Z_1 is obtained by considering the portion of span a.

$$Z_1 = \frac{Ma}{2} \times \frac{a}{3} \times \frac{1}{EI}$$

$$= \frac{Ma^2}{6EI}$$

Also Z_2 is found by considering the whole of the M diagram

$$Z_2 = \left\{\frac{Mb^2}{2} + \frac{Ma}{2}\left(\frac{a}{3} + b\right)\right\}\frac{1}{EI}$$

$$= \left\{\frac{Mb^2}{2} + \frac{Ma^2}{6} + \frac{Mab}{2}\right\}\frac{1}{EI}$$

$$\therefore Z = \frac{1}{EI}\left\{\frac{Mb^2}{2} + \frac{Ma^2}{6} + \frac{Mab}{2} - \frac{Ma^2}{6} - \frac{Mab}{6}\right\}$$

$$= \frac{1}{EI}\left\{\frac{Mb^2}{2} + \frac{Mab}{3}\right\}$$

$$\text{or } Z = \frac{Mb}{6EI}\{2a + 3b\} \text{ where } M = Pa$$

80. **Horizontal Beam with uniform load supported at each end, and having a prop at mid-span at same level as end supports.**

If the centre prop is removed the deflection at mid-span is $Z = \frac{5}{384}\frac{WL^3}{EI}$. The upward force P at mid-span, required to bring the mid-point of the beam back to the same height as the end supports, is given by $\frac{PL^3}{48EI}$ = upward deflection = Z,

$$\therefore \frac{PL^3}{48EI} = \frac{5}{384}\frac{WL^3}{EI}$$

$$\text{and } P = \tfrac{5}{8} W$$

$\therefore$ Each end support carries $\frac{3}{16}$ W.

81. **Cantilever with a Uniform Load and having a *rigid* prop at the free end.**

On removing the prop the deflection Z at the free end becomes $\frac{1}{8}\frac{WL^3}{EI}$. The single upward force P at the free end required to destroy the deflection Z is given by $Z = \frac{1}{3}\frac{PL^3}{EI}$

$$\therefore \frac{1}{3}\frac{PL^3}{EI} = \frac{1}{8}\frac{WL^3}{EI}$$

$$\text{or} \quad P = \tfrac{3}{8} W$$

82. Sinking Props.

If the props in (80) and (81) sink below the level of the horizontal, so that the deflection removed is $\frac{1}{p}$ of the deflection when no prop is in position, then the value of P will be $\frac{1}{p}$ that given in (80) and (81).

83. Elastic Prop.

Suppose the prop supporting the cantilever, as in par. 81, to be elastic. Then if P is the force exerted by the prop, E_p Young's modulus for the material of the prop, and L_p its length, we have that the shortening of the prop $= \frac{PL_p}{A_pE_p}$ where A_p is the cross-sectional area of the prop. This shortening of the prop will be equal to the difference between the free deflection and the upward deflection caused by P.

$$\therefore \frac{PL_p}{A_pE_p} = \frac{WL^3}{8EI} - \frac{PL^3}{3EI}$$

$$\therefore \quad P = \frac{\frac{WL^3}{8EI}}{\frac{L_p}{A_pE_p} + \frac{L^3}{3EI}}$$

84. Beams of Uniform Strength.

The bending equation gives $f = \frac{My}{I}$. The condition, therefore, of uniform strength is that $\frac{My}{I}$ shall be constant; that is, the section modulus shall be proportional to the bending moment. The value of $\frac{I}{y}$ may be varied, in the case of rectangular beams, by altering the depth or altering the breadth.

(*a*) *Constant depth.*

Taking the case of a beam supported at each end, and considering one-half of the M diagram, the maximum deflection is obtained from

$$Z = \int_0^{\frac{L}{2}} \frac{M}{EI} x \, dx$$

Let M_1 be the bending moment at mid-span, and I_1 the moment of inertia at mid-span, and M and I the bending

moment and moment of inertia at a point distant x from one end. Since the depth is constant

$$\frac{M_1}{I_1} = \frac{M}{I}$$

$$\therefore \quad Z = \frac{M_1}{I_1 E}\int_0^{\frac{L}{2}} x dx \quad . \quad . \quad . \quad . \quad (1)$$

$$= \frac{M_1 L^2}{8EI_1} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For an isolated load at mid-span $M_1 = \frac{WL}{4}$ and

$$\therefore \quad Z = \frac{WL^3}{32EI_1} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a uniformly distributed load $M_1 = \frac{WL}{8}$ and

$$Z = \frac{WL^3}{64EI_1} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The deflection of a cantilever may be obtained by substituting L for $\frac{L}{2}$ in (1) when $Z = \frac{M_1 L^2}{2EI_1}$, I_1 being the moment of inertia at the constraint, and M_1 the bending moment at the same point.

For a cantilever with a uniform load $M_1 = \frac{WL}{2}$ and

$$Z = \frac{WL^3}{4EI_1} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For a cantilever with a concentrated load at the free end $M_1 = WL$

$$\text{and } Z = \frac{WL^3}{2EI_1} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(*b*) *Constant breadth.*

$$f = \frac{M_1 y_1}{I_1} = \frac{My}{I} \text{ or } \frac{M_1}{\frac{1}{6}bD_1^2} = \frac{M}{\frac{1}{6}bd^2}$$

$$\therefore \quad \frac{D_1}{d} = \sqrt{\frac{M_1}{M}}$$

$$\text{and } \frac{M}{I} = \frac{M_1}{I_1}\frac{D_1}{d} = \frac{M_1}{I_1}\sqrt{\frac{M_1}{M}}$$

where D_1 is the depth at mid-span and d the depth at a point distant x from the support. In the case of a beam resting on supports, then, as before, considering one-half the span

$$Z = \int_0^{\frac{L}{2}} \frac{M}{EI} x \, dx$$

$$= \frac{M_1^{\frac{3}{2}}}{EI_1} \int_0^{\frac{L}{2}} M^{-\frac{1}{2}} x \, . \, dx \quad . \quad . \quad . \quad (7)$$

For a concentrated load at mid-span $M = \frac{W}{2} x$ and $M_1 = \frac{WL}{4}$

$$\therefore \quad Z = \frac{WL^3}{24EI_1} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Other cases may be solved by substituting the value of M in (7) and integrating.

EXAMPLE 4.

A steel girder having a uniform depth of 12 ft. rests on piers 120 ft. apart, and carries a uniformly distributed load. Find the deflection at the centre, (*a*) when the girder is of uniform cross-section, and the maximum flange stress is 6½ tons/sq. in., (*b*) when the area of the flanges is so proportioned that there is a uniform flange stress of 6½ tons/sq. in. E = 13,500 tons/sq. in. (Lond. Univ., 1918.)

$$(a)\ Z = \frac{5}{384} \frac{WL^3}{EI_1}$$

but $M_1 = \frac{WL}{8}$ where M_1 and I_1 refer to mid-span.

$$\therefore Z = \frac{5}{384} \times \frac{8M_1L^2}{EI_1} \text{ and } \frac{M_1}{I_1} = \frac{f}{y}$$

$$= \frac{5}{48} \frac{L^2}{E} \times \frac{f}{y}$$

$$= \frac{5}{48} \times \frac{(120)^2 \times 144}{13{,}500} \times \frac{6{\cdot}5}{72}$$

$$= 1{\cdot}445 \text{ in.}$$

$$(b)\ Z = \frac{M_1 L^2}{8EI_1}$$

$$= \frac{L^2}{8E} \times \frac{f}{y}$$

$$= \frac{(120)^2 \times 144}{8 \times 13{,}500} \times \frac{6\cdot 5}{72}$$

$$= 1\cdot 734 \text{ in.}$$

85. **Deflection of a Built-up Spring.** This spring consists of a number of curved plates (Fig. 67), each of constant width b, and thickness t, and is so designed that each plate has the same maximum bending stress at all cross-sections. The load is applied at mid-span, and is resisted by reactions at each end, and that load which causes all the plates to straighten is called the proof load.

FIG. 67

Assuming each plate to be an arc of a circle, and that all the plates have the same curvature, then the change of curvature will be the same for each plate.

Considering the section at mid-span, the bending moment is $\frac{WL}{4}$, and, since each plate has the same maximum stress, the resisting moment of each plate will be $\frac{WL}{4N}$ where N is the number of plates. If f is the maximum stress, we have

$$= \frac{M}{I} y = \frac{WL}{4N} \times \frac{12t}{2\,(bt^3)}$$

$$= \frac{3}{2} \frac{WL}{Nbt^2} \qquad \qquad (1)$$

The deflection may be obtained by considering the longest plate, since each is an arc of a circle.

$$Z = \frac{ML^2}{8EI} \quad \text{par. 79 } (h)$$

$$= \frac{WL}{32N} \times \frac{L^2}{EI} = \frac{WL^3}{32NE} \times \frac{12}{bt^3}$$

$$= \frac{3}{8} \frac{WL^3}{NEbt^3} \qquad \qquad (2)$$

Since the maximum stress f is to be the same at all transverse sections $\frac{My}{I}$ must be constant, and since the breadth of each plate, b, is constant, then the depth of the spring must be varied. The number of plates, therefore, at any transverse section must be proportional to the bending moment at that section. The ends of the plates usually have their width tapered in order that the resisting moment may have the same rate of change.

EXAMPLE 5.

A carriage spring of the ordinary type, 30 in. long, is made of steel plates each $2\frac{1}{2}$ in. wide and $\frac{1}{4}$ in. thick. How many plates are required, if the central load is 800 lb., and if the maximum stress in the steel is not to exceed 12 tons/sq. in.? What will be the deflection under the above load, if $E = 30{,}000{,}000$ lb./sq. in. Prove the correctness of any formula you employ in determining the number of plates which are required. (Lond. Univ., 1914.)

$$f = \frac{3}{2}\frac{WL}{Nbt^2}$$

$$\therefore \quad N = \frac{3}{2}\frac{WL}{fbt^2}$$

$$= \frac{3 \times 800 \times 30 \times 2 \times 16}{2 \times 12 \times 2240 \times 5}$$

$$= 8{\cdot}55$$

Thus 9 plates would be used.

$$Z = \frac{3}{8}\frac{WL^3}{NEbt^3}$$

$$= \frac{3}{8} \times \frac{800 \times 30 \times 30 \times 30 \times 2 \times 64}{9 \times 30 \times 10^6 \times 5}$$

$$= 0{\cdot}77 \text{ in.}$$

86. **Mathematical Treatment.** Fig. 68 represents a beam deflected by the application of a variable bending moment. Consider the *small* portion of the beam of length ds whose horizontal distance from a fixed point is x, and vertical distance from the same point, y. The slope of the beam over ds is given

by $\frac{dy}{dx}$, also if θ is the angle which the tangent to the curve makes with the axis of x, then

$$\frac{dy}{dx} = \tan\theta, \frac{ds}{dx} = \sec\theta \text{ and } \frac{d\theta}{ds} = \frac{1}{R}$$

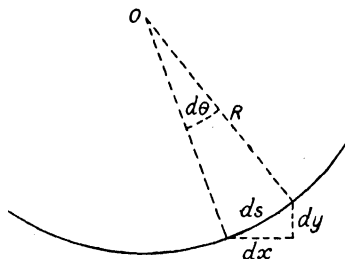



FIG. 68

$$\therefore \frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$

$$= \sec^2\theta \frac{d\theta}{ds} \cdot \frac{ds}{dx}$$

$$= \sec^3\theta \frac{1}{R}$$

$$\therefore \quad \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\sec^3\theta}$$

$$\text{also } \sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \quad \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

In the beams met with in practice $\frac{dy}{dx}$ is very small, and may be neglected when raised to higher powers.

$$\therefore \frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\text{but } \frac{1}{R} = \frac{M}{EI}$$

where M is the bending moment on ds

$$\therefore \frac{d^2y}{dx^2} = \frac{M}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{The slope of the beam} = \frac{dy}{dx} = \int \frac{M}{EI}\, dx \quad . \quad . \quad . \quad (2)$$

$$\text{The deflection } y = \int \frac{dy}{dx}\, dx = \iint \frac{M}{EI}\, dx\, dx \quad . \quad (3)$$

(*a*) Horizontal beam freely supported at each end, with an isolated load W at mid-span.

Taking the origin at mid-span, and measuring distances x *out* from mid-span, and y from a tangent at the point of maximum deflection *up* to the beam.

$$EI\frac{d^2y}{dx^2} = M$$

$$= \frac{W}{2}\left(\frac{L}{2} - x\right)$$

$$\therefore EI\frac{dy}{dx} = \frac{W}{2}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) + c$$

$$\text{at } x = 0 \; \frac{dy}{dx} = 0 \quad \therefore c = 0$$

$$\therefore \quad EIy = \frac{W}{2}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) + c_1$$

$$\text{at} \quad x = 0,\, y = 0 \quad \therefore c_1 = 0$$

Thus the slope is found from

$$\frac{dy}{dx} = \frac{W}{2EI}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) \quad . \quad . \quad . \quad . \quad (1)$$

The deflection at any point is given by

$Z - y$ where Z is the maximum deflection and

$$y = \frac{W}{2EI}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) \quad . \quad . \quad . \quad (2)$$

at $\quad x = \frac{L}{2}, y = Z$

$$\therefore \quad Z = \frac{W}{2EI}\left(\frac{L^3}{16} - \frac{L^3}{48}\right)$$

$$= \frac{WL^3}{48EI} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*b*) Horizontal beam freely supported at ends and carrying a uniformly distributed load w per unit run.

Taking the origin and other measurements as in *a*,

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) - \frac{w}{2}\left(\frac{L}{2} - x\right)^2$$

$$= \frac{wL}{2}\left(\frac{L}{2} - x\right) - \frac{w}{2}\left(\frac{L}{2} - x\right)^2$$

$$= \frac{w}{2}\left(\frac{L^2}{4} - x^2\right)$$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(\frac{L^2x}{4} - \frac{x^3}{3}\right) + c$$

at $\quad x = 0 \; \frac{dy}{dx} = 0$ and $\therefore c = 0$

$$\therefore \quad EIy = \frac{w}{2}\left(\frac{L^2x^2}{8} - \frac{x^4}{12}\right) + c_1$$

at $\quad x = 0, y = 0,$ and $\therefore c_1 = 0$

Thus slope is given by

$$\frac{dy}{dx} = \frac{w}{2EI}\left(\frac{L^2x}{4} - \frac{x^3}{3}\right) \quad . \quad . \quad . \quad (4)$$

and deflection by $Z - y$ where Z is the maximum deflection.

$$\text{and} \qquad y = \frac{w}{2EI}\left(\frac{L^2x^2}{8} - \frac{x^4}{12}\right) \quad . \quad . \quad (5)$$

$$\text{at} \qquad x = \frac{L}{2}, \; y = Z$$

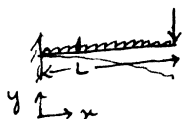
$$\therefore \qquad Z = \frac{w}{2EI}\left(\frac{L^4}{32} - \frac{L^4}{192}\right)$$

$$= \frac{5}{384}\,\frac{wL^4}{EI}$$

$$= \frac{5}{384}\,\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(c) Cantilever with span L and isolated load at a distance from the fixed end.

Taking the origin at the fixed end, and measuring y downwards from the horizontal to the beam, y is now the deflection.

$$EI\,\frac{d^2y}{dx^2} = W\,(l - x)$$

$$EI\,\frac{dy}{dx} = W\left(lx - \frac{x^2}{2}\right) + c$$

$$\text{at } x = 0 \;\; \frac{dy}{dx} = 0 \quad \therefore c = 0$$

$$\text{and} \qquad EIy = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + c_1$$

$$\textit{at } x = 0, \; y = 0 \qquad \therefore c_1 = 0$$

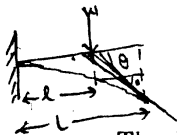
Thus the slope is given by

$$\frac{dx}{dy} = \frac{W}{EI}\left(lx - \frac{x^2}{2}\right) . \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{and the deflection by } y = \frac{W}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \quad . \quad . \quad . \quad (8)$$

The deflection at W is found by substituting $x = l$ in above equation

$$\therefore y_W = \frac{W}{EI}\left(\frac{l^3}{2} - \frac{l^3}{6}\right)$$

$$= \frac{1}{3}\frac{Wl^3}{EI} \qquad \ldots \ldots \quad (9)$$

The deflection at the free end = deflection at W + slope at $W\,(L - l)$

$$= \frac{1}{3}\frac{Wl^3}{EI} + (L - l)\left(\frac{W}{EI}\left(l^2 - \frac{l^2}{2}\right)\right)$$

$$= \frac{W}{EI}\left\{\frac{l^3}{3} + \frac{Ll^2}{2} - \frac{l^3}{2}\right\}$$

$$= \frac{W}{EI}\left\{\frac{Ll^2}{2} - \frac{l^3}{6}\right\} \qquad \ldots \ldots \quad (10)$$

(*d*) Cantilever with isolated load at free end.

The slope and deflection are obtained by substituting L for l in (7) and (8) respectively.

The maximum deflection $= \dfrac{1}{3}\dfrac{WL^3}{EI} \qquad \ldots \ldots \quad (11)$

(*e*) Cantilever with uniformly distributed load.

Taking the origin, etc., as in (*c*),

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(L - x)^2$$

$$= \frac{w}{2}(L^2 - 2\,Lx + x^2)$$

$$EI\frac{dy}{dx} = \frac{w}{2}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) + c$$

at $\qquad x = 0 \quad \dfrac{dy}{dx} = 0 \quad \therefore c = 0$

$$\text{and } EIy = \frac{w}{2}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) + c_1$$

at $\qquad x = 0,\ y = 0 \quad \therefore c_1 = 0.$

Slope is thus given by

$$\frac{dy}{dx} = \frac{w}{2EI}\left(L^2x - Lx^2 + \frac{x^3}{3}\right) \quad . \quad . \quad . \quad . \quad (12)$$

and deflection by $\frac{w}{2EI}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right) \quad . \quad . \quad . \quad (13)$

The maximum deflection is obtained when $x = L$ and

$$= \frac{wL^4}{2EI}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right)$$

$$= \frac{1}{8}\frac{wL^4}{EI}$$

$$= \frac{1}{8}\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

(*f*) Cantilever with uniform load only over a distance l from fixed end.

From 12*e* the slope at end of l is

$$\frac{w}{2EI}\left(\frac{l^3}{3}\right)$$

$$= \frac{Wl^2}{6EI}$$

and the deflection at end of $l = \frac{1}{8}\frac{Wl^3}{EI}$

∴ Deflection at free end

$$= \text{deflection at end of } l + (L - l) \text{ slope at end of } l$$

$$= \frac{1}{8}\frac{Wl^3}{EI} + (L - l)\left(\frac{Wl^2}{6EI}\right)$$

$$= \frac{W}{EI}\left(\frac{l^3}{8} + \frac{Ll^2}{6} - \frac{l^3}{6}\right)$$

$$= \frac{W}{EI}\left(\frac{Ll^2}{6} - \frac{l^3}{24}\right) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

(*g*) Horizontal beam freely supported at its ends, with an isolated load not at mid-span.

We have $R_A = \dfrac{Wb}{a+b}$ and $R_B = \dfrac{Wa}{a+b}$ (Fig. 69).

Taking A as the origin and measuring x and y as shown
From $x = 0$ to $x = a$ we have

$$EI\frac{d^2y}{dx^2} = R_B(L - x) - W(a - x)$$

$$\therefore \quad EI\frac{dy}{dx} = R_B\left(Lx - \frac{x^2}{2}\right) + \frac{W(a - x)^2}{2} + c \qquad . \qquad . \quad (1)$$

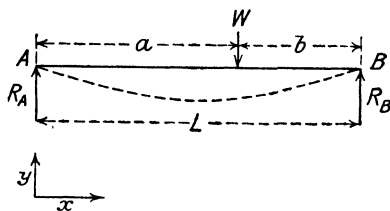



FIG. 69

From $x = a$ to $x = L$ we have

$$EI\frac{d^2y}{dx^2} = R_B(L - x)$$

$$\text{and} \quad EI\frac{dy}{dx} = R_B\left(Lx - \frac{x^2}{2}\right) + c_1 \qquad . \quad . \quad . \quad . \quad (2)$$

(1) and (2) must agree at $x = a$. $\therefore$ c_1 must equal c.
Integrating (1),

$$EIy = R_B\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{W(a - x)^3}{6} + cx + c_2 \qquad . \quad (3)$$

and integrating (2),

$$EIy = R_B\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + c_1x + c_3 \qquad . \quad . \quad . \quad (4)$$

These must give the same value of y at $x = a$,

$\therefore \quad c_2 = c_3.$

At $x = 0$, $y = 0$, therefore from (3) $c_2 = \dfrac{Wa^3}{6}$

substituting in (4)

$$EIy = R_B\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + c_1x + \frac{Wa^3}{6}$$

at $x = L,\ y = 0$

$$\therefore \quad c_1 = -R_B\left(\frac{L^2}{2} - \frac{L^2}{6}\right) - \frac{Wa^3}{6L}$$

$$= -R_B\frac{L^2}{3} - \frac{Wa^3}{6L} \quad . \quad . \quad . \quad (5)$$

Substituting the values of c and c_2 in (3)

$$EIy = R_B\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{W(a-x)^3}{6} - R_B\frac{L^2x}{3} - \frac{Wa^3x}{6L} + \frac{Wa^3}{6} \quad (6)$$

$$EIy = \frac{Wa}{a+b}\left(\frac{3a^3 + 3a^2b - a^3}{6}\right) - \frac{Wa}{a+b}\left(\frac{2a^3 + 4a^2b + 2ab^2}{6}\right)$$

$$-\frac{Wa}{a+b}\left(\frac{a^3 - a^3 - a^2b}{6}\right) = \frac{Wa}{a+b} \times \frac{-2ab^2}{6}$$

$\therefore$ Deflection under the load

$$= \frac{1}{3}\frac{Wa^2b^2}{(a+b)EI} \quad . \quad . \quad . \quad . \quad (7)$$

The maximum deflection is obtained when $\frac{dy}{dx} = 0$

$\therefore$ from (1)

$$R_B\left(Lx - \frac{x^2}{2}\right) + \frac{W(a-x)^2}{2} - R_B\left(\frac{L^2}{3}\right) - \frac{Wa^3}{6L} = 0$$

$$R_B\left(Lx - \frac{x^2}{2} - \frac{L^3}{3}\right) + \frac{W}{L}\left\{\frac{(a-x)^2}{2}L - \frac{a^3}{6}\right\} = 0$$

which gives $\quad x^2 = \frac{1}{3}a(a + 2b) \quad . \quad . \quad . \quad . \quad (8)$

and by substitution in (6), the maximum deflection is given by—

$$\frac{Wb(a^2 + 2ab)^{\frac{3}{2}}}{9\sqrt{3}EI(a+b)} \quad . \quad . \quad . \quad (9)$$

87. **Deflection Due to Shear.** A further deflection occurs in beams owing to the shearing forces on transverse sections. This deflection may be found approximately by making use of the

equation for shear stress at a point in the transverse section of a beam, which is, in itself, based on the assumption that pure bending occurs.

(a) Cantilever with load at free end.

Assuming the section to be rectangular of breadth b and depth d, and the total length of the beam L. If Z_s is the deflection, due to shear, at the free end, then the work done by the load $= \frac{1}{2}WZ_s$ . . . . . . (1)

The shear stress at a distance y from the neutral axis is by (8), par. 75.

$$q = \frac{6W}{bd^3}\left(\frac{d^2}{4} - y^2\right) \text{ where } F = W$$

also, if dy is the height of the strip in the direction of the depth of the beam, and we consider a small portion of the beam of length dx, and section $b \times dy$, we have by (3), par. 28, that the strain energy in the strip is

$$\frac{1}{2}\frac{q^2}{C}\,dx\,bdy$$

$$\text{or strain energy} = \frac{1}{2C}\,dx \cdot \frac{36W^2}{b^2d^6}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right) bdy$$

$\therefore$ Total strain energy for piece of beam of length dx

$$= \frac{18W^2dx}{bd^6C}\int_{-\frac{d}{2}}^{+\frac{d}{2}}\left(\frac{d^4}{16} - \frac{d^2y^2}{2} + y^4\right) dy$$

$$= \frac{36W^2dx}{bd^6C}\left|\frac{d^4y}{16} - \frac{d^2y^3}{6} + \frac{y^5}{5}\right|_0^{\frac{d}{2}}$$

$$= \frac{3}{5}\frac{W^2dx}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Strain energy for whole beam of length L

$$= \frac{3}{5}\frac{W^2}{bdC}\int_0^L dx$$

$$= \frac{3}{5}\frac{W^2L}{bdC} \quad . \quad . \quad . \quad . \quad (3)$$

Equating the strain energy to the work done by W

$$\frac{1}{2} W Z_s = \frac{3}{5} \frac{W^2 L}{bdC}$$

$$\therefore \qquad Z_s = \frac{6}{5} \frac{WL}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the total deflection at the free end due to bending and shear is $Z = Z_b + Z_s$

$$= \frac{1}{3} \frac{WL^3}{EI} + \frac{6}{5} \frac{WL}{bdC} \quad . \quad . \quad . \quad (5)$$

(*b*) Horizontal beam with isolated load at mid-span.

If we treat each half of the span as a cantilever the L in (4) becomes $\frac{L}{2}$, and W becomes $\frac{W}{2}$

and $$Z_s = \frac{3}{10} \frac{WL}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \text{ Total deflection} = \frac{1}{48} \frac{WL^3}{EI} + \frac{3}{10} \frac{WL}{bdC} \quad . \quad . \quad . \quad (7)$$

(*c*) Cantilever with uniformly distributed load.

In *a* the shearing force F is constant along the beam, but for a uniformly loaded cantilever at distance x from the fixed end, the shearing force is $w(L-x)$, hence (2) becomes

$$\frac{3}{5} \frac{w^2(L-x)^2}{bdC} dx$$

$\therefore$ Strain energy for beam

$$= \frac{3w^2}{5bdC} \int_0^L (L-x)^2 dx$$

$$= \frac{3}{5} \frac{w^2}{bdC} \left| L^2 x - Lx^2 + \frac{x^3}{3} \right|_0^L$$

$$= \frac{1}{5} \frac{w^2 L^3}{bdC}$$

$$= \frac{1}{5} \frac{W^2 L}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\therefore \qquad Z_s = \frac{2}{5}\frac{WL}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\text{Total deflection} = \frac{1}{8}\frac{WL^3}{EI} + \frac{2}{5}\frac{WL}{bdC} \quad . \quad . \quad . \quad (10)$$

(*d*) Horizontal beam with uniformly distributed load.

Treating each half of the span as a cantilever, L becomes $\frac{L}{2}$ in (9) and W becomes $\frac{W}{2}$.

$$\therefore \qquad Z_s = \frac{1}{10}\frac{WL}{bdC} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\text{Total deflection} = \frac{5}{384}\frac{WL^3}{EI} + \frac{1}{10}\frac{WL}{bdC} \quad . \quad . \quad . \quad (12)$$

For the majority of cases of beams, except those used for the main girder of a bridge, etc., the ratio of $\frac{d}{L}$ is so large that the deflection due to shear is negligible in comparison to that due to bending.

88. **Deflection Due to an Impact.** Suppose a load W to impinge on a beam of span L, supported freely at its ends, at a point half-way along the span. If h is the height fallen through by W, and δ the resultant deflection produced, then the work done by W is $W(h + \delta)$, provided the resultant stress is not beyond the limit of elasticity of the beam.

If W_1 is the equivalent static load at mid-span to produce the deflection δ, the work done by W_1 is given by $\frac{W_1 \times \delta}{2}$. The strain energy produced in the beam is the same in each case

$$\therefore \qquad \tfrac{1}{2}W_1\delta = W(h + \delta) \quad . \quad . \quad . \quad . \quad (1)$$

but
$$\delta = \frac{1}{48}\frac{W_1L^3}{EI}$$

hence
$$\frac{1}{96}\frac{W_1^2L^3}{EI} = W\left(h + \frac{1}{48}\frac{W_1L^3}{EI}\right)$$

or
$$W_1^2\frac{L^3}{96EI} - W_1\frac{L^3W}{48EI} - Wh = 0$$

$$\therefore \qquad W_1^2 - 2W_1 . W - \frac{96WhEI}{L^3} = 0$$

and hence
$$W_1 = W + \sqrt{W^2 + \frac{96WhEI}{L^3}} \quad . \quad . \quad (2)$$

the value of δ is found by substituting (2) in equation (1).

88A. **Deflection of a Framework.** If a number of external forces act on a framework, then the work done by these forces is equal to the strain energy of the framework.

This equation enables us to determine the deflection at a point on a loaded framework.

Let F = force acting in a bar

f = stress in bar due to F

A = cross-sectional area of the bar

l = length of bar

P = the *single* external load acting at a joint

x = the deflection of the joint in the direction of P

The strain energy of the bar $= \frac{f^2}{2E} \times$ volume of bar

$$= \frac{f^2}{2E} Al$$

The strain energy of the f× 10work

$$= \sum \frac{f^2}{2E} Al \quad . \quad . \quad . \quad (1)$$

$$= \tfrac{1}{2}\sum \frac{F^2l}{AE} \quad . \quad . \quad . \quad (2)$$

The external work done by $P = \tfrac{1}{2}Px$ . . . . (3)

then, by equating (2) and (3)

$$\tfrac{1}{2}Px = \tfrac{1}{2}\sum \frac{F^2l}{AE}$$

$$\therefore x = \frac{1}{P}\sum \frac{F^2l}{AE} \quad . \quad . \quad . \quad (4)$$

The forces acting in the bars are usually found by drawing a force diagram.

When more than one load acts on a framework, the deflection at a joint may be found as follows—

Fig. 69A shows a framework acted on by a load system P_1 P_2 P_3 P_4 and P_5, and the vertical deflection of the joint K is required.

Let F denote the force in any bar due to the above load system, and x the deflection at K. Now introduce the load Q

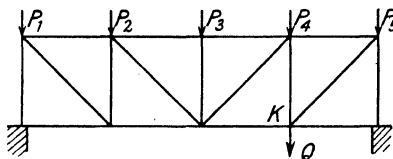



FIG. 69A

at K in the direction in which the deflection is required, and let F_1 denote the force in the bar due to Q, and x_1 the deflection at K due to Q.

The total force in the bar is $F_2 = F + F_1$ and the total deflection at K is $x_2 = x + x_1$

$$\text{The strain energy of the bar } = \tfrac{1}{2}F_2 \frac{F_2 l}{AE} \quad . \quad . \quad . \quad (5)$$

$$\text{and the portion due to} \quad Q = \tfrac{1}{2}F_1 \frac{F_2 l}{AE}$$

The total strain energy due to

$$Q = \tfrac{1}{2}\Sigma F_1 \frac{F_2 l}{AE} \quad . \quad . \quad . \quad (6)$$

$$\text{The external work due to } Q = \tfrac{1}{2}Qx_2$$

$$= \tfrac{1}{2}Q(x + x_1) \quad . \quad . \quad . \quad (7)$$

$$\therefore \quad \tfrac{1}{2}Q(x + x_1) = \tfrac{1}{2}\Sigma F_1 \frac{F_2 l}{AE}$$

$$x + x_1 = \Sigma \frac{F_1}{Q} \cdot \frac{F_2 l}{AE}$$

$$= \Sigma \frac{F_1}{Q} \frac{Fl}{AE} + \Sigma \frac{F_1}{Q} \frac{F_1 l}{AE} \quad . \quad (8)$$

$$\text{by (4) } x_1 = \frac{1}{Q}\sum\frac{F_1^{\,2}l}{AE} \text{ and hence } x = \sum\frac{F_1}{Q}\frac{Fl}{AE} \quad . \quad (9)$$

usually most convenient to take Q as a unit force. The force in each bar is found for the given load system and then for unit load only acting at K. The value of F and the ratio $\frac{F_1}{Q}$ is then substituted in (9).

EXAMPLE 6.

The wall crane shown in Fig. 69B has the following dimensions. The bar CD is 10 ft. long and 1 sq. in. in cross-section, the remaining bars are each

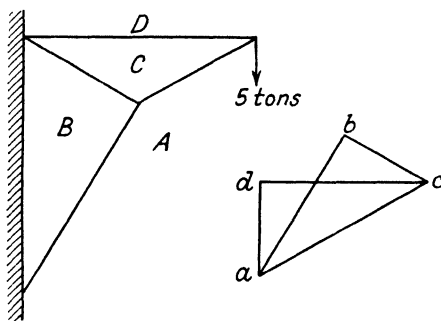



FIG. 69B

3 sq. in. in cross-section. Determine the vertical deflection at the point of application of the load. $E = 30 \times 10^6$ lb./sq. in.

The diagram is drawn to scale and the length of each bar measured, then from the force diagram the force in each bar is obtained.

Bar	Length (inches)	Force (tons)	Cross-sectional area sq. in.	$\frac{F^2l}{A}$
CD	120	8·66	1	9,000
CA	69·4	10·00	3	2,313
CB	69·4	5·00	3	578
BA	120	8·66	3	3,000

$$\sum\frac{F^2l}{A} = 14{,}891$$

$$\therefore \quad x = \frac{1}{P}\sum\frac{F^2 l}{AE}$$

$$= \frac{14{,}891 \times 2240}{5 \times 30 \times 10^6}$$

$$= 0{\cdot}222 \text{ in.}$$

EXAMPLE 7.

Draw the force diagram for the truss shown in Fig. 69c, and find the deflection at the point of attachment of the central 10 ton load. All members are 12 ft. long and 6 sq. in. in cross-section. $E = 13{,}500$ tons/sq. in. (Lond. Univ., 1935.)

The force diagram shown gives the magnitude and nature of the force F in each bar, tension being taken as positive and compression as negative. Unit load Q is then assumed to act

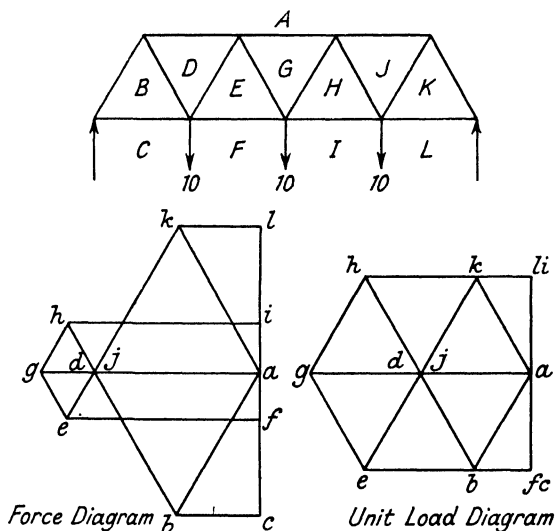



FIG. 69c

at the point where the deflection is required and a new diagram drawn. This diagram gives the force F_1 in each bar due to this loading.

Since the structure and the loading are symmetrical in each case, it would be sufficient to draw one half of each diagram. The diagrams are drawn to different scales, the unit load diagram being drawn to a large scale for accuracy.

Bar	F (tons)	F_1 (tons)	$\frac{F_1}{Q}F$
AD	−17·3	−0·575	9·950
AG	−23·1	−1·15	26·550
AJ	−17·3	−0·575	9·950
KL	+ 8·65	+ 0·289	2·475
IH	+ 20·2	+ 0·868	17·700
EF	+ 20·2	+ 0·868	17·700
BC	+ 8·65	+ 0·289	2·475
AB	−17·3	−0·575	9·950
BD	+ 17·3	+ 0·575	9·950
DE	−5·8	−0·575	3·325
EG	+ 5·8	+ 0·575	3·325
GH	+ 5·8	+ 0·575	3·325
HJ	−5·8	−0·575	3·325
JK	+ 17·3	+ 0·575	9·950
AK	−17·3	−0·575	9·950

$$\Sigma\frac{F_1}{Q}F = 139{\cdot}9$$

Since A l and E are the same for all bars, then

$$\text{Deflection} = \frac{139{\cdot}9 \times 12 \times 12}{6 \times 13{,}500}$$
$$= 0{\cdot}249 \text{ in.}$$

EXAMPLES VIII

1. A vertical mild steel column 6 in. diameter is securely bedded into the ground. The total height of the column is 12 ft. The upper end is subjected to a horizontal load of 3,000 lb. Calculate the maximum stress at the ground section and the deflection at the top. Modulus of elasticity 30,000,000 lb./sq. in. (Lond. Univ., 1918.) *Ans.*, 20,300 lb./sq. in., 1·565 in.

2. A cantilever has a free length of 20 ft., and the moment of inertia of its cross-section about the principal axis is 81·1 (inches)^4. If a couple of 6 tons ft. is applied to the free end, find the deflection (*a*) at the free end, (*b*) mid-way between the free end and the support. Take Young's modulus for the material to be 13,000 tons/sq. in. (A.M.I.Mech.E., 1925.) *Ans.*, 1·96 in., 0·49 in.

3. A beam of uniform section of length l and weight W, is built rigidly into a wall at one end and is supported on a column at the other end, the two supports being originally at the same level. The column supporting the free end is elastic, the force necessary to cause unit compression in it being F. Prove that, when the beam has settled into position, the load carried by the support is $W\left[\frac{3}{8\left(1+\frac{3EI}{l^3F}\right)}\right]$, E and I having their usual significance. (Lond. Univ., 1915.)

4. A horizontal rolled steel joist 10 in. × 6 in. is supported at its ends and has a span of 10 ft. A load of 400 lb. falls from a height of 3½ in. on to the middle of the joist. Neglecting the loss of energy at impact, find the maximum instantaneous stress produced in the joist, given that the moment of inertia of the section is 210 in. units and $E = 13{,}250$ tons/sq. in. (Lond. Univ., 1922.) *Ans.*, 7·1 tons/sq. in.

5. A strip of brass 3 in. × ¾ in. × 10 ft. long, and weighing 76 lb., is observed to have a slight curvature. It is supported at its ends with the 3 in. width horizontal and the sag in the middle is observed to be 1·64 in. When turned completely over the sag is 1·13 in. Find the sag due to the original curvature of the strip and determine the modulus of elasticity of the brass. Obtain any formula you use. (Lond. Univ., 1923.)
Ans., Sag = 0·255 in., $E = 11{,}700{,}000$ lb./sq. in.

6. A beam of constant I-section is supported freely at each end and covers a span of 16 ft. It carries a concentrated load of 12 tons at 6 ft. from the left-hand support. The moment of inertia of any section of the beam is 212 (inch)^4 units. Find the deflection under the load. Prove the formula you employ. The modulus of elasticity, E, of the material of the beam, is 13,500 tons/sq. in. (Lond. Univ., 1919.) *Ans.*, 0·5436 in.

7. A cantilever is loaded along its length of 6 ft. at 1 cwt. per foot run close to the support, increasing uniformly to 7 cwt./ft. run over the free end. Obtain the shape of the cantilever if it is of wood 3 in. breadth at all points and to be equally stressed all along its top edge at 500 lb./sq. in. Take $E = 1{\cdot}5 \times 10^6$ lb./sq. in. What would be the slope in degrees at the free end? (Lond. Univ., 1918.) *Ans.*, Depth at constraint = 21·94 in., at mid-span 12 in.

8. A cast-iron girder has a clear span of 20 ft. and is supported at the ends, the top flange is 4 in. × 1½ in., bottom flange 12 in. × 2 in., and web 14½ in. × 1½ in. Calculate the safe distributed load for a limiting stress of 1½ tons/sq. in. tension and 8 tons/sq. in. compression. What will be the deflection? $E = 6{,}000$ tons/sq. in. (A.M.I.Mech.E., 1919.) *Ans.*, 15·5 tons and 0·236 in.

9. A beam of depth d carries a total uniformly distributed load of W across a span L; the beam being simply supported. Assuming the maximum deflection limited to $\frac{1}{480}$ span, show that L must not exceed $20d$ if the stress is limited to 8 tons/sq. in. $E = 13{,}500$ tons/sq. in. (A.M.I.Mech.E., 1918.)

10. A straight beam 10 ft. long is supported on two supports 7 ft. apart, so that one end of the beam is on one support, and loaded with a uniformly distributed load of 1 ton per foot run. The moment of inertia of the beam section is 300 (inch)^4 units. Determine (1) the slope of the beam at both of its ends, (2) the deflection relative to the fixed supports at a point half-way between the supports and also at the unsupported end of the beam. (Lond. Univ., 1923.)

Ans., $\dfrac{4{\cdot}4}{E}$, $\dfrac{0{\cdot}34}{E}$ radians, $\dfrac{101}{E}$ in. and $\dfrac{7{\cdot}2}{E}$ in.

11. A beam of constant cross-section and 10 ft. long is freely supported at its ends. It is loaded at points 3 ft. from each end with load of 1 ton. Find the ratio of the deflection under the centre of the beam to the deflection at a point under one of the loads. (Lond. Univ., 1921.) *Ans.*, 1·22.

12. A uniform beam of length $l + 2a$ is supported at two points l ft. apart and overhung at each end a length a. It is loaded with concentrated loads W at each extremity and $2W$ at the centre of the span. Determine expressions for the deflection at each load. (Lond. Univ., 1916.)

Ans., At end $\dfrac{Wa^3}{3EI} + \dfrac{Wa^2l}{2EI} - \dfrac{Wal^2}{8EI}$, at mid-span $\dfrac{Wl^2}{8EI}\left(\dfrac{l}{3} - a\right)$

13. A motor-car spring of the laminated plate type is 36 in. long and is made up of plates each 3 in. wide by $\frac{3}{8}$ in. thick. How many plates are required if the central load is 0·5 ton, and the maximum stress is not to exceed 15 tons/sq. in.? $E = 30,000,000$ lb. sq. in. Estimate the total deflection and write down the assumptions made in obtaining this estimate. (Lond. Univ., 1922.) *Ans.*, 5 plates, 0·84 in.

14. A laminated steel carriage spring 30 in. long is built up of 9 leaves of equal thickness and 2 in. wide. Neglecting friction, find the thickness of the leaves if the stress in the material is to be limited to 25 tons/sq. in. when the spring is loaded with 750 lb. in the centre.

To what radius should the leaves be initially bent for the spring to be flat when under this load and what will be its central deflection? $E = 30 \times 10^6$ lb./sq. in. (Lond. Univ., 1930.)

Ans., 0·183 in., 49 in., and 2·3 in.

15. A cantilever has a prop P at a distance L from the fixed end and on this length there is a uniformly distributed load w tons/ft. run. The cantilever projects a distance $\frac{L}{4}$ beyond the prop and on this length there is a uniformly distributed load of $2w$ tons/ft. run. If the prop is rigid and holds its point of application on the horizontal, find what proportion of the total load W is taken by the prop. If $w = 0·6$ tons/ft. run and $L = 20$ ft., draw to scale the shear force and bending moment diagrams. (Lond. Univ., 1935.)

Ans., $\frac{31}{48}W$.

16. A cantilever of length L carries a uniformly distributed load w tons/ft. run. A rigid prop is inserted at a distance l from the fixed end, and the prop rests at the mid point of a freely supported beam of length x. Find x in terms of l if the deflection of the cantilever at the prop is one-third of its value with the prop removed. Assume E and I to be the same for both beams. Where should the prop be placed for it to carry half the total load on the cantilever and at the same time fulfil the above conditions regarding the deflection? (Lond. Univ., 1930.)

Ans., $x = 2l$; $l = 0·645L$.

17. Two similar cantilevers are placed one above the other a distance x apart and their free ends are joined by an elastic rod of diameter d. A load W is placed at mid span on the lower cantilever, and the moment of inertia and span of each is I and L respectively. Show that the pull in the rod is given by $\dfrac{5WL^3}{32\left(\dfrac{6xI}{\pi d^2} + L^3\right)}$

18. A beam of length l is freely supported at its ends and carries two concentrated loads W at distances $l/3$ from each end. Find the ratio of the deflection of the beam midway between the supports to that which would be produced by a single load W placed midway between the supports. (I.Mech.E., 1936.)

Ans., $\frac{46}{27}$

19. A crane jib 20 ft. long and 10 sq. in. in cross-sectional area is attached to a rigid support 10 ft. vertically below the end of the tie-rod. The tie-rod is 13 ft. long and has a cross-sectional area of 4 sq. in. Find the elastic horizontal and vertical deflections of the crane head when a load of 7 tons is applied to

it, whose line of action makes 30° to the right of a vertical drawn through the crane head. Assume that the modulus of direct elasticity = 29×10^6 lb./sq. in. (Lond. Univ., 1931.)

Ans., 0·0845 in. and 0·146 in. respectively.

20. The truss shown in Fig. 69D is loaded uniformly, and the sections are such that all compression members are stressed to 4 tons/sq. in., and the

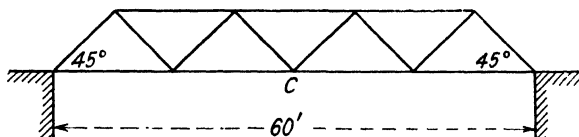



FIG. 69D

tension members to 5 tons/sq. in. Find the deflection of the centre point C. E = 13,000 tons/sq. in. (Lond. Univ., 1930.)

Ans., 0·745 in.

21. A horizontal beam has each end placed between pegs which exert no constraint on the beam. A couple M is applied in the plane of the beam at a point distant a from one support and b from the other support. Show that the deflection of the beam at the point of application of the couple is given by—

$$\delta = \frac{Mab\,(a - b)}{3EI\,(a + b)}$$

23. A horizontal beam carries a load W placed at a distance a from one end and a distance b from the other end. The ends of the beam are supported by cantilevers of length d and c, the former being continuous with the portion a of the beam and the latter continuous with that of the portion b. Show that if E and I are the same for all the beams then the deflection under the load is given by—

$$\delta = \frac{W}{3EIl^2}\,(b^2d^3 - a^2c^3 + a^2b^2l)$$

where l is equal to $a + b$.

CHAPTER IX

FIXED AND CONTINUOUS BEAMS

89. **Fixed Beams.** When the ends of a beam are constrained to remain in a horizontal position, the beam is said to be "fixed," "encastré," or "built in." The slope of the beam is thus zero at each end, and a couple will have to be applied at each end to make the slope there have this value. The applied couples will be of opposite sign to that of the bending moment, due to the loading, and thus points of contra flexure

FIG. 70

will occur as shown by Fig. 70 at A and B. Since at each end of the beam there is a bending moment, the shape of the M and S curves will be different from those of a corresponding beam simply supported at each end. Comparatively few cases of fixed beams exist in practice, and it should be remembered that where a beam enters a wall, for a small portion of its length, it is not constrained sufficiently to treat it as fixed.

We will now consider several cases of fixed beams, treating them firstly from a knowledge of the area of the M diagram, and secondly from a mathematical standpoint.

We have $\theta = \frac{A}{EI}$ for a beam of uniform section, but $\theta = 0$ for a fixed beam, at each constraint, and since EI is not zero, then A, *the resultant area of the M diagram for the beam, must be zero.*

(*a*) *Fixed beam with isolated central load.*

If the beam were simply supported, the M diagram would be represented by ABC (Fig. 71), but since the ends are fixed, and the load is at mid-span, there will be a couple of magnitude EA applied at each end, which has a sign opposite to that of ABC. By combining the positive and negative M diagrams,

the resultant diagram $AEFBGDC$ is obtained. Since area $ABC + EACD = 0$

$$EA \times L = \frac{WL}{4} \times \frac{L}{2}$$

$$\text{Couple} = EA = \frac{WL}{8}$$

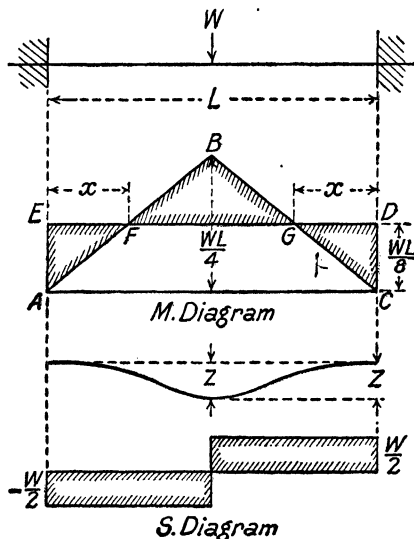



FIG. 71

and the points of contraflexure will be at G and F distant x from each end, where $x = \frac{L}{4}$.

The value of M at mid-span is equal to that of M at each end $= \frac{WL}{8}$. The beam is, therefore, twice as strong as a simply-supported beam.

Considering half the span, and taking a chosen line through one of the constraints, we have the deflection at mid-span given by—

$$Z = \frac{A\bar{x}}{EI}$$

$$= \frac{1}{EI}\left(\frac{WL}{4} \times \frac{L}{4} \times \frac{2}{3}\frac{L}{2} - \frac{WL}{8} \times \frac{L}{2} \times \frac{L}{4}\right)$$

$$= \frac{WL^3}{EI}\left(\frac{1}{48} - \frac{1}{64}\right)$$

$$= \frac{1}{192}\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

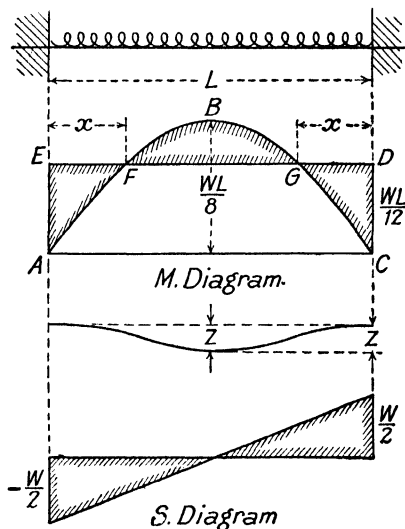



FIG. 72

Thus, the stiffness of the beam is four times that of a similar beam, simply supported.

The shearing force at each end $= \frac{W}{2}$, and is uniform up to mid-span, where it changes sign and continues uniform and equal to $-\frac{W}{2}$. The S diagram is thus similar to that for a freely-supported beam.

(*b*) *Horizontal beam with uniformly distributed load.*

Using the same reasoning as in (*a*), we have

$$EA \,.\, L = \frac{2}{3}\frac{WL}{8} \times L \text{ (Fig. 72)}$$

$$EA = \frac{1}{12} WL$$

and the bending moment at mid-span $= \frac{WL}{8} - \frac{WL}{12} = \frac{1}{24} WL$.

To find the distance x of the points of contraflexure from each end, we have

$$\frac{WL}{8} - \frac{w\left(2\left(\frac{L}{2} - x\right)\right)^2}{8} = \frac{WL}{12}$$

$$\frac{w\left(\frac{L}{2} - x\right)^2}{2} = \frac{WL}{24} = \frac{wL^2}{24}$$

$$\left(\frac{L}{2} - x\right)^2 = \frac{L^2}{12}$$

$$\frac{L}{2} - x = \frac{L}{2\sqrt{3}}$$

$$x = 0{\cdot}211L$$

The deflection Z at mid-span is obtained from

$$Z = \frac{\left(\frac{WL}{8} \times \frac{L}{2} \times \frac{2}{3} \times \frac{5L}{8 \times 2}\right) - \left(\frac{WL}{12} \times \frac{L}{2} \times \frac{L}{4}\right)}{EI}$$

$$= \frac{WL^3}{EI}\left(\frac{5}{384} - \frac{1}{96}\right)$$

$$= \frac{1}{384} \frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The shearing force at each end is $\frac{W}{2}$, and decreases at a uniform rate. It is zero at mid-span, then it increases uniformly to $-\frac{W}{2}$. The S diagram is thus similar to that for a simply-supported beam. It will be noted that the above beam is five times as stiff, and one-and-a-half times as strong, as a simply-supported beam.

EXAMPLE 1.

A horizontal beam has a clear span of 20 ft., and its two ends are rigidly built in. Calculate the shearing force and bending moment at the two ends and at the centre, when supporting a wall, which is equivalent to a uniformly distributed load of 1·5 ton per foot run. Also determine the position of the points of inflection, and make a sketch showing the distribution of bending moment and shearing force for the beam. (A.M.I.Mech.E., 1919.)

The bending moment at each end

$$= \frac{1}{12} WL$$

$$= \frac{1}{12} \times 20 \times 1{\cdot}5 \times 20 \times 12$$

$$= 600 \text{ ton in.}$$

The bending moment at the centre

$$= \frac{1}{24} WL$$

$$= 300 \text{ ton in.}$$

The shearing force at each end

$$= \frac{W}{2}$$

$$= \frac{20 \times 1{\cdot}5}{2}$$

$$= 15 \text{ tons}$$

The shearing force at the centre $= 0$

The distance of the points of inflection from each end

$$= 0{\cdot}211L$$

$$= 0{\cdot}211 \times 20$$

$$= 4{\cdot}22 \text{ ft.}$$

The bending moment and shearing force diagrams are similar to those in Fig. 72.

90. **Effect of Fixing on the M Diagram.** Fig. 73 represents a fixed beam without any external applied load. At the ends A and B couples of magnitude M_A and M_B are applied respectively, the shearing forces at A and B being S_A and S_B.

Taking moments about A

$$M_A = M_B - S_B L$$

$$\therefore \qquad S_B = \frac{M_B - M_A}{L} \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Similarly} \qquad S_A = \frac{M_A - M_B}{L} \quad . \quad . \quad . \quad . \quad (2)$$

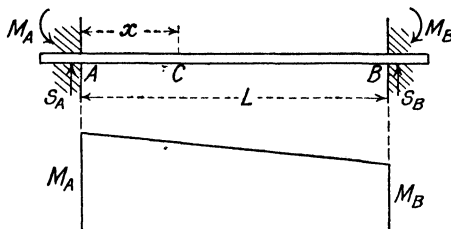



FIG. 73

The bending moment at a point C, distant x from A, is found from

$$M_x = M_B - S_B (L - x)$$

$$= M_B + \frac{M_A - M_B}{L}(L - x) \quad . \quad . \quad (3)$$

or

$$M_x = M_A - S_A (x)$$

$$= M_A + \frac{M_B - M_A}{L} x \quad . \quad . \quad . \quad (4)$$

Thus, from (3) or (4), it will be seen that the M diagram slopes at a uniform rate from one end to the other.

91. **Fixed Beam with any Loading.** Referring to Fig. 74, let A_1 be the area of the free bending diagram $AEFHB$, and let $\bar{x}_1$ be the distance of its centroid from A, also let A_2 be the area of $ACDB$, and $\bar{x}_2$ the distance of its centroid from A.

Then

$$A_1 = A_2$$

and

$$= \frac{(M_A + M_B)}{2} L$$

or

$$M_A + M_B = \frac{2A_1}{L} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also

$$\left(\frac{A_1\bar{x}_1 + (-A_2\bar{x}_2)}{EI}\right) = 0$$

since each end of the beam remains horizontal.

$$\therefore \qquad A_1x_1 = A_2\bar{x}_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and since $\qquad A_1 = A_2$

then $\qquad \bar{x}_1 = \bar{x}_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

FIG. 74

Taking moments about A we have

$$A_2\bar{x}_2 = \frac{M_BL^2}{2} + \frac{M_A - M_B}{2}L \times \frac{L}{3}$$

$$= \frac{M_BL^2}{3} + \frac{M_AL^2}{6} \quad . \quad . \quad . \quad (4)$$

or

$$A_1\bar{x}_1 = \frac{M_BL^2}{3} + \frac{M_AL^2}{6} \quad . \quad . \quad . \quad (5)$$

hence

$$2M_B + M_A = \frac{6A_1\bar{x}_1}{L^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From (1) and (6) we get

$$M_A = \frac{4A_1}{L} - \frac{6A_1\bar{x}_1}{L^2} \quad . \quad . \quad . \quad . \quad (7)$$

and

$$M_B = \frac{6A_1\bar{x}_1}{L^2} - \frac{2A_1}{L} \quad . \quad . \quad . \quad . \quad (8)$$

92. Fixed Beam with Isolated Load not at Mid-span. The maximum ordinate of the free bending-moment diagram (Fig. 75) is $\frac{Wab}{(a+b)}$, and from (7) and (8), par. 91

$$M_A = \frac{4}{(a+b)}\left(\frac{Wab}{(a+b)} \times \frac{(a+b)}{2}\right) - \frac{6}{(a+b)^2}$$
$$\left\{\frac{Wab}{(a+b)} \cdot \frac{a}{2} \cdot \frac{2a}{3} + \frac{Wab}{(a+b)} \cdot \frac{b}{2}\left(a + \frac{b}{3}\right)\right\}$$
$$= \frac{Wab^2}{(a+b)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

FIG. 75

and $$M_B = \frac{6}{(a+b)^2}\left\{\frac{Wab}{(a+b)} \cdot \frac{a}{2} \cdot \frac{2a}{3} \cdot + \frac{Wab}{(a+b)} \cdot \frac{b}{2} \cdot \left(a + \frac{b}{3}\right)\right\}$$
$$- \frac{2}{a+b}\left\{\frac{Wab}{(a+b)} \frac{(a+b)}{2}\right\}$$
$$= \frac{Wa^2b}{(a+b)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking moments about B

$$R_A(a+b) = M_A - M_B + Wb$$
$$= \frac{Wab}{(a+b)^2}(b-a) + Wb$$
$$= \frac{Wb}{(a+b)^2}\{ab - a^2 + a^2 + 2ab + b^2\}$$

$$\therefore \qquad R_A = \frac{Wb^2(3a+b)}{(a+b)^3} \quad . \quad . \quad . \quad . \quad (3)$$

Taking moments about A

$$R_B(a+b) = M_B - M_A + Wa$$
$$= \frac{Wab}{(a+b)^2}(a-b) + Wa$$
$$= \frac{Wa}{(a+b)^2}(ab - b^2 + a^2 + 2ab + b^2)$$

and
$$R_B = \frac{Wa^2(a+3b)}{(a+b)^3} \quad . \quad . \quad . \quad . \quad (3A)$$

The distance of the points of inflection from the ends of the beam is found from

$$\frac{x_1}{a} = \frac{M_A + \frac{(M_B - M_A)}{(a+b)}x_1}{\frac{Wab}{a+b}}$$

which gives
$$x_1 = \frac{a(a+b)}{3a+b} \quad . \quad . \quad . \quad . \quad (4)$$

and
$$\frac{x_2}{b} = \frac{M_B - \frac{(M_B - M_A)}{a+b}x_2}{\frac{Wab}{(a+b)}}$$

which gives
$$x_2 = \frac{b(a+b)}{a+3b} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The negative bending moment under the load is equal to

$$\frac{Wab^2}{(a+b)^2} + a\frac{Wab}{(a+b)^2}\cdot\frac{a-b}{a+b}$$
$$= \frac{Wab}{(a+b)^2}\frac{(a^2+b^2)}{a+b}$$

and the deflection under the load is given by—

$$Z = \left[\frac{Wab}{(a+b)}\frac{b}{2}\cdot\frac{2b}{3} - \frac{Wab}{(a+b)^2}\frac{(a^2+b^2)}{(a+b)}\frac{b}{2}\cdot\frac{2b}{3} - \frac{Wa^2b}{(a+b)^2}\frac{b}{2}\cdot\frac{b}{3}\right.$$
$$\left. - b\left\{\frac{Wab}{(a+b)}\frac{b}{2} - \frac{Wab}{(a+b)^2}\frac{(a^2+b^2)}{a+b}\frac{b}{2} - \frac{Wa^2b}{(a+b)^2}\frac{b}{2}\right\}\right]\frac{1}{EI}$$
$$= \frac{1}{3}\frac{Wa^3b^3}{(a+b)^3EI} \quad . \quad . \quad . \quad . \quad (6)$$

The resultant bending moment at E, distant x from A, and between A and the load, is

$$\frac{Wb}{(a+b)}x - \frac{Wab^2}{(a+b)^2} - \frac{Wab}{(a+b)^2}\frac{(a-b)}{(a+b)}x \quad . \quad . \quad . \quad (7)$$

The slope at E

$$= \frac{1}{EI}\left\{\frac{Wb}{(a+b)}\frac{x^2}{2} - \frac{Wab^2}{(a+b)^2}x - \frac{Wab}{(a+b)^2}\frac{(a-b)}{(a+b)}\frac{x^2}{2}\right\} \quad . \quad (8)$$

The maximum deflection will occur where (8) is zero, i.e. when

$$\frac{Wb}{(a+b)}\frac{x^2}{2} - \frac{Wab^2}{(a+b)^2}x - \frac{Wab}{(a+b)^2}\frac{(a-b)}{(a+b)}\frac{x^2}{2} = 0$$

or

$$x = \frac{2a(a+b)}{3a+b} \quad . \quad . \quad . \quad . \quad (9)$$

and the maximum deflection is found by substituting the value of x in (9) in

$$Z = \left[\frac{Wb}{(a+b)}\frac{x^3}{3} - \frac{Wab^2}{(a+b)^2}\frac{x^2}{2} - \frac{Wab}{(a+b)^2}\left(\frac{a+b}{a-b}\right)\frac{x^3}{3}\right.$$

$$\left. - x\left\{\frac{Wb}{(a+b)}\frac{x^2}{2} - \frac{Wab^2}{(a+b)^2}x - \frac{Wab}{(a+b)^2}\left(\frac{a-b}{a+b}\right)\frac{x^2}{2}\right\}\right]\frac{I}{EI} \quad (10)$$

which gives

$$Z_{max} = \frac{2}{3}\frac{Wa^3b^2}{(3a+b)^2EI} \quad . \quad . \quad . \quad (11)$$

93. **Fixed Beam Ends not at Same Level.** Uniformly distributed load w per unit run.

Let Z be the difference in level between the ends of the beam (Fig. 76). A point of contraflexure occurs at mid-span, and each half of the beam might be taken as a cantilever, the free end of which is caused to deflect $\frac{Z}{2}$ by a load H at the free end.

Then

$$\frac{Z}{2} = \frac{1}{3}\frac{H(\frac{1}{2}L)^3}{EI}$$

$$\therefore \quad H = \frac{12EIZ}{L^3}$$

The bending moment diagram on each half, due to H, is triangular, the maximum ordinate being

$$\pm \frac{12EIZ}{L^3} \times \frac{L}{2} = \pm \frac{6EIZ}{L^2}$$

The fixing bending moments will be altered, due to the difference in level of A and B by the above amount, and hence this

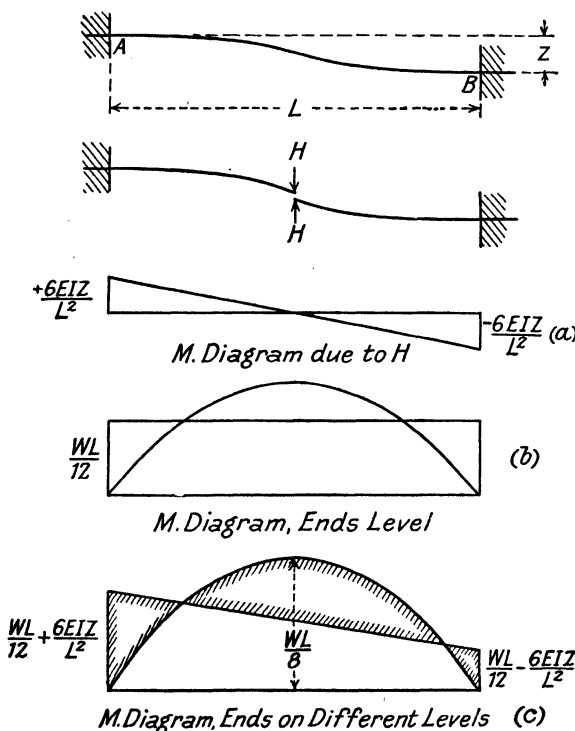



Fig. 76

quantity must be added to, or subtracted from, M_A and M_B, depending on which side is the higher. When A is above B,

$$M_A = \frac{WL}{12} + \frac{6EIZ}{L^2} \text{ and } M_B = \frac{WL}{12} - \frac{6EIZ}{L^2}.$$

The combined diagram of bending moment due to H, and to the fixing of the ends, is shown at (c), Fig. 76.

94. Mathematical Treatment of Fixed Beams. (*a*) *Fixed beam with isolated load at mid-span.*

Since the beam is symmetrical, the fixing moment is equal in magnitude at each end, and equal to, say, M_F. Referring to Fig. 71, we have that from $x = 0$ to $x = \frac{L}{2}$, where origin is taken at midspan,

$$EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) - M_F$$

and
$$EI\frac{dy}{dx} = \frac{W}{2}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) - M_F x + C$$

at
$$x = 0, \frac{dy}{dx} = 0, \therefore C = 0.$$

$\therefore$
$$EI\frac{dy}{dx} = \frac{W}{2}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) - M_F x \quad . \quad . \quad . \quad (1)$$

also
$$EIy = \frac{W}{2}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) - \frac{M_F x^2}{2} + c_1$$

at
$$x = \frac{L}{2}, \frac{dy}{dx} = 0$$

$\therefore$
$$M_F = \frac{WL}{8} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

at $x = 0, y = 0, \therefore c_1 = 0$

and
$$EIy = \frac{W}{2}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) - \frac{WL}{8}\frac{x^2}{2} \quad . \quad . \quad (3)$$

at $x = \frac{L}{2}$ the deflection is maximum.

$$EIy = \frac{W}{2}\left(\frac{L}{4} \times \frac{L^2}{4} - \frac{1}{6}\frac{L^3}{8}\right) - \frac{WL}{8} \times \frac{L^2}{4} \times \frac{1}{2}$$

$$= \frac{1}{192} WL^3$$

$\therefore$
$$y = \frac{1}{192}\frac{WL^3}{EI} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The bending moment at mid-span $= \frac{W}{2}\left(\frac{L}{2} - 0\right) - \frac{WL}{8} = \frac{WL}{8}$.

(*b*) *Fixed beam with uniformly increasing load.*

Referring to (3), par. 59, the free bending moment at the point C is $\frac{fL^2}{6}x - \frac{fx^3}{6}$, also, by (4), par. 90, the negative bending moment at C is $M_A + \frac{M_B - M_A}{L}x$.

$\therefore$ Resultant bending moment at C

$$= \frac{fL^2}{6}x - \frac{fx^3}{6} - M_A - \frac{M_B - M_A}{L}x$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{fL^2x^2}{12} - \frac{fx^4}{24} - M_Ax - \frac{M_B - M_A}{L}\frac{x^2}{2} + C$$

at $x = 0$, $\frac{dy}{dx} = 0$, $\therefore C = 0$, and at $x = L$, $\frac{dy}{dx} = 0$

$$\therefore \quad \frac{fL^4}{12} - \frac{fL^4}{24} - M_AL - \frac{M_B - M_A}{L}\frac{L^2}{2} = 0$$

$$\text{or} \quad M_A + M_B = \frac{fL^3}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{also} \quad EIy = \frac{fL^2x^3}{36} - \frac{fx^5}{120} - \frac{M_Ax^2}{2} - \frac{M_B - M_A}{L}\frac{x^3}{6} + C_1$$

at $x = 0$, $y = 0$, $\therefore C_1 = 0$, and at $x = L$, $y = 0$

$$\therefore \quad \frac{fL^5}{36} - \frac{fL^5}{120} - \frac{M_AL^2}{2} - \frac{M_B - M_A}{L}\frac{L^3}{6} = 0$$

$$\text{or} \quad M_B + 2M_A = \frac{7}{60}fL^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from (1) and (2)

$$M_B = \frac{1}{20}fL^3 = \frac{WL}{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$M_A = \frac{1}{30}fL^3 = \frac{WL}{15} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The bending moment diagram is shown by Fig. 77.

(c) *Horizontal beam, fixed at one end, supported at the other end, and carrying a central load W.*

Let P be the supporting force at the free end, L the length of the beam, and M_F the fixing couple at the fixed end. Taking the fixed end as origin, we have

$$EI\frac{d^2y}{dx^2} = P(L-x) - W\left(\frac{L}{2}-x\right) \quad . \quad . \quad . \quad (1)$$

from $x = 0$ to $x = \frac{L}{2}$

and $$EI\frac{d^2y}{dx^2} = P(L-x) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

from $x = \frac{L}{2}$ to $x = L$.

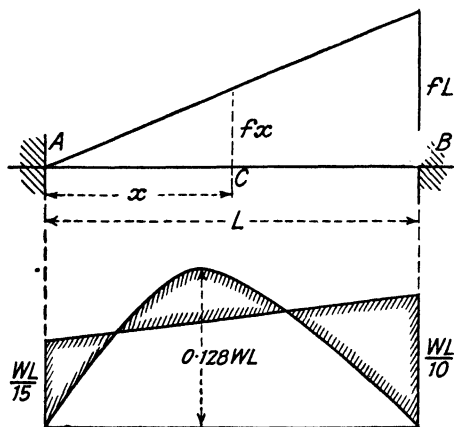



FIG. 77

$$\text{From (1) } EI\frac{dy}{dx} = P\left(Lx - \frac{x^2}{2}\right) + W\frac{\left(\frac{L}{2}-x\right)^2}{2} + C \quad . \quad (3)$$

$$\text{,, (2) } EI\frac{dy}{dx} = P\left(Lx - \frac{x^2}{2}\right) + C_1 \quad . \quad . \quad . \quad (4)$$

(3) and (4) must agree at $x = \frac{L}{2}$.

$$\therefore \qquad C = C_1$$

at $x = 0,\ \frac{dy}{dx} = 0,\ \therefore$ from (3) $C = C_1 = -\frac{WL^2}{8}$

$$EIy = P\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - W\frac{\left(\frac{L}{2} - x\right)^3}{6} - \frac{WL^2}{8}x + C_2 \quad . \quad (5)$$

and

$$EIy = P\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{WL^2}{8}x + C_3 \quad . \quad . \quad . \quad . \quad (6)$$

(5) and (6) must agree at $x = \frac{L}{2}$. $\therefore C_2 = C_3$.

At $x = 0$, $y = 0$, and at $x = L$, $y = 0$.

$\therefore$ from (5), $\frac{WL^3}{48} = C_2 = C_3$.

And substituting in (6) when $x = L$,

$$0 = P\left(\frac{L^3}{2} - \frac{L^3}{6}\right) - \frac{WL^3}{8} + \frac{WL^3}{48}$$

$$\therefore P = \frac{5}{16}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Taking moments about the fixed end

$$PL - \frac{WL}{2} + M_F = 0$$

$$\therefore \qquad M_F = \frac{3}{16}WL \quad . \quad . \quad . \quad . \quad (8)$$

Problems of this type may also be solved by treating them as continuous beams (see later examples).

95. **Disadvantages of Fixed Beams.** At first sight, it would appear to be a decided advantage to use a fixed beam in preference to a simply supported beam, since the fixed beam is much stiffer and stronger, also the maximum bending moment often occurs at the fixing, and thus the strengthening of the beam to take this bending moment does not increase its weight to any great extent, as would occur in a simply supported beam. Referring to par. 93, it will be observed that, if the ends are not at the same level, then the bending moment is greatly increased, hence any settling of the supports of a fixed beam would cause a great alteration in the stresses. In a fixed beam also the slope at each end must be zero, and any variation of this will cause alteration in the estimated stresses.

These two factors, which go to make the actual stresses uncertain, are the main reasons for not adopting fixed beams universally. If a fixed beam be hinged at the points of contra-flexure, then each end portion forms a cantilever, and the centre portion a simply supported beam. This method obviates

Resultant M. Diagram

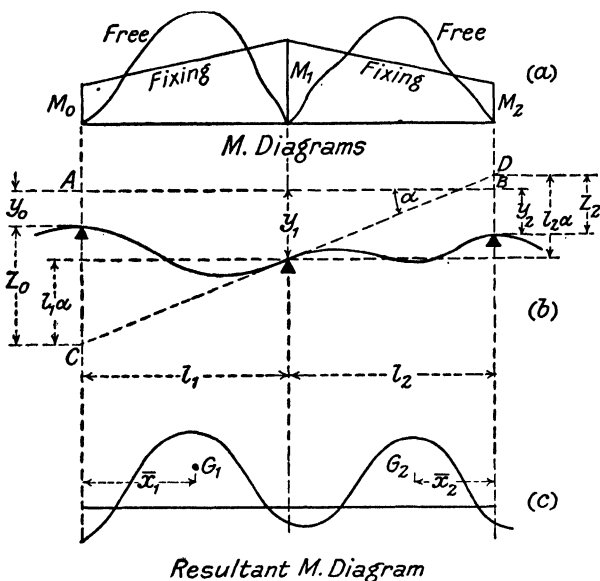



FIG. 78

the above disadvantages, and is utilized in the construction of cantilever bridges.

96. **Continuous Beams.** A beam resting on more than two supports is said to be continuous. Such a beam is represented by Fig. 78 (*b*). Changes of curvature occur in each span, due to negative bending moments at the supports. In the case represented the supports are assumed to be at different levels, being distances y_0, y_1 and y_2 from a horizontal line AB. Suppose the loading to be such that the "free" and "fixing" bending moment diagrams are as shown at (*a*), the resultant diagram being shown at (*c*). The area of the resultant diagram on the span l_1 is A_1, and the distance of its centroid from a vertical line through the left-hand support is $\bar{x}_1$; also the area of the resultant diagram on the span l_2 is A_2, and $\bar{x}_2$ is the

distance of its centroid from a vertical line through the right-hand support.

Draw CD, a common tangent at the point of contact of the central support, and let a be its inclination to the horizontal, agreeing to take intercepts, between tangents to the deflected beam, on a vertical line as positive when measured downwards, and vice versa as negative, we have

$$z_o = \frac{A_1\bar{x}_1}{EI} = y_1 - y_o + l_1 a \quad . \quad . \quad (1)$$

$$-z_2 = \frac{A_2 x_2}{EI} = -l_2 a + (y_1 - y_2) \quad . \quad . \quad (2)$$

and $$\frac{A_1\bar{x}_1}{EIl_1} - \frac{y_1 - y_o}{l_1} = a = -\frac{A_2\bar{x}_2}{EIl_2} + \frac{y_1 - y_2}{l_2}$$

$$\therefore \quad \frac{A_1\bar{x}_1}{EIl_1} + \frac{A_2\bar{x}_2}{EIl_2} = \frac{y_1 - y_o}{l_1} + \frac{y_1 - y_2}{l_2}$$

and $$\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2} = \left\{\frac{y_1 - y_o}{l_1} + \frac{y_1 - y_2}{l_2}\right\}EI \quad . \quad . \quad (3)$$

When the supports are all at the same level $y_o = y_1 = y_2$

and $$\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If the areas of the "free" bending moment diagrams are S_1 and S_2, and the distance of the centroid of S_1 from a vertical line through the left-hand support is x_1, the corresponding distance of the centroid of S_2 from a vertical through the right-hand support is x_2, then from (4)

$$\frac{1}{l_1}\left\{S_1x_1 - M_o l_1 \frac{l_1}{2} - \frac{M_1 - M_o}{2} l_1 \times \frac{2l_1}{3}\right\}$$
$$+ \frac{1}{l_2}\left\{S_2x_2 - M_2 l_2 \frac{l_2}{2} - \frac{M_1 - M_2}{2} l_2 \times \frac{2l_2}{3}\right\} = 0$$

or $$\frac{S_1x_1}{l_1} + \frac{S_2x_2}{l_2} - \frac{M_o l_1}{6} - \frac{M_1}{3}(l_1 + l_2) - \frac{M_2 l_2}{6} = 0$$

and $$M_o l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = 6\left\{\frac{S_1x_1}{l_1} + \frac{S_2x_2}{l_2}\right\} \quad . \quad (5)$$

This is called "Clapeyron's Theorem of Three Moments," and with its aid, by taking the spans in pairs, sufficient equations are obtained to solve for all the "fixing" bending moments.

97. **Shearing Forces and Reactions.** Consider the spans l_1 and l_2 of the continuous beam (Fig. 79), and let the reactions at BC and D be R_B, R_C, and R_D respectively. If the shearing force on each side of the support C be F_C and $F^1{}_C$, then $R_C = F_C + F^1{}_C$.

Let W_1 be the total load on l_1 and $\bar{x}_B$ the horizontal distance of its centroid from B, and let W_2 be the total load on l_2, the horizontal distance of its centroid from D being $\bar{x}_D$.

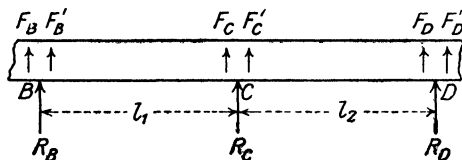



FIG. 79

Considering the span BC, and taking moments about B

$$M_B - M_C + F_C l_1 = W_1 \bar{x}_B$$

or

$$F_C = \frac{M_C - M_B}{l_1} + \frac{W_1 \bar{x}_B}{l_1} \quad . \quad . \quad . \quad . \quad (1)$$

Considering span CD, and taking moments about D

$$M_D - M_C + F_C{}^1 l_2 = W_2 \bar{x}_D$$

or

$$F_C{}^1 = \frac{M_C - M_D}{l_2} + \frac{W_2 \bar{x}_D}{l_2} \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad R_C = F_C + F^1{}_C$$

$$= \frac{M_C - M_B}{l_1} + \frac{M_C - M_D}{l_2} + \frac{W_1 \bar{x}_B}{l_1} + \frac{W_2 \bar{x}_D}{l_2} \quad (3)$$

If the loading on l_1 and l_2 be w_1 and w_2 per unit run, then

$$\frac{W_1 \bar{x}_B}{l_1} = \frac{w_1 l_1}{2} \text{ and } \frac{W_2 \bar{x}_D}{l_2} = \frac{w_2 l_2}{2}$$

and

$$R_C = \frac{M_C - M_B}{l_1} + \frac{M_C - M_D}{l_2} + \frac{w_1 l_1}{2} + \frac{w_2 l_2}{2} \quad . \quad (4)$$

The reaction and shearing forces at the other supports are found by a similar method.

EXAMPLE 2.

Draw the bending moment and shearing force diagrams for a continuous beam which is supported at three points at the same level, but free at its extremities. The spans are 50 ft. and 35 ft.; the 50 ft. span supports two loads of value 2,000 lb. and 1,000 lb., distant 20 ft. and 40 ft. respectively from the free end, and the 35 ft. span is loaded uniformly with 100 lb. per foot run.

The maximum ordinate of the free bending moment diagram for the 35 ft. span

$$= \frac{WL}{8} = \frac{100 \times 35 \times 35 \times 12}{8}$$

$$= 183{,}750 \text{ lb. in.}$$

In the free bending moment diagram for the 50 ft. span, the bending moment at the 2000 lb. load

$$= 1400 \times 240$$

$$= 336{,}000 \text{ lb. in.}$$

The bending moment at the 1000 lb. load

$$= (1400 \times 480) - (2000 \times 240)$$

$$= 192{,}000 \text{ lb. in.}$$

Referring to Fig. 80, $M_A = 0$ and $M_C = 0$, since the ends are free. Then the equation—

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = 6\left\{\frac{S_1 x_1}{l_1} + \frac{S_2 x_2}{l_2}\right\}$$

becomes

$$2M_B(l_1 + l_2) = 6\left\{\frac{S_1 x_1}{l_1} + \frac{S_2 x_2}{l_2}\right\}$$

$$2M_B(50 + 35)\,12$$

$$= 6\left\{\frac{\left(\frac{336{,}000 \times 240}{2} \times \frac{40 \times 12}{3}\right) + (192{,}000 \times 240 \times 30 \times 12}{600}\right.$$

$$+ \frac{\left(\frac{144{,}000 \times 240}{2} \times \frac{80}{3} \times 12\right) + \left(\frac{192{,}000 \times 120}{2} \times 520\right)}{600}$$

$$\left. + \frac{\left(\frac{183{,}750 \times 420 \times 2}{3} \times \frac{420}{2}\right)}{420}\right\}$$

$$\therefore \qquad M_B = 244{,}500 \text{ lb. in.}$$

$$S_A = 1400 + \left(\frac{0 - 244{,}500}{600}\right) = 1400 - 408 = 992 \text{ lb.}$$

$$S_A - 2000 = 992 - 2000 = -1008 \text{ lb.}$$

$$S_B = S_A - 2000 - 1000 = -1008 - 1000 = -2008 \text{ lb.}$$

$$S^1_B = \frac{3500 \times \frac{35 \times 12}{2}}{35 \times 12} + \frac{244{,}500 - 0}{420}$$

$$= 1750 + 582$$

$$= 2332 \text{ lb.}$$

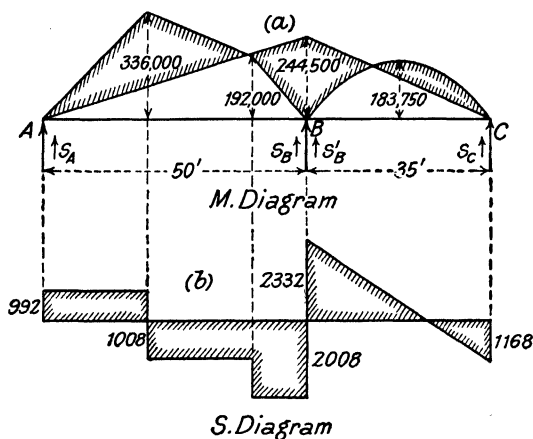



Fig. 80

$$S_C = -\left\{1750 + \frac{0 - 244{,}500}{420}\right\}.$$

$$= -\{1750 - 582\}$$

$$= -1168 \text{ lb.}$$

The shearing force diagram is shown at Fig. 80.

98. **Beams Fixed at One End.** When a beam is fixed at one end, and propped at the other end, the theorem of Three Moments may be applied to find the bending moments, by imagining a span of exactly similar dimensions and similar

loading to exist beyond the other side of the constraint. The following examples will illustrate the method of attack.

EXAMPLE 3.

A cantilever 20 ft. long is propped at one end, and a load of 1,000 lb. is placed at mid-span. Draw the bending moment and shearing force diagrams, and find the distance of the point of contraflexure from the constraint. (Compare par. 94 (c).)

Referring to Fig. 81, suppose another similar span to exist to the left of B, and let A_1 be its propped end.

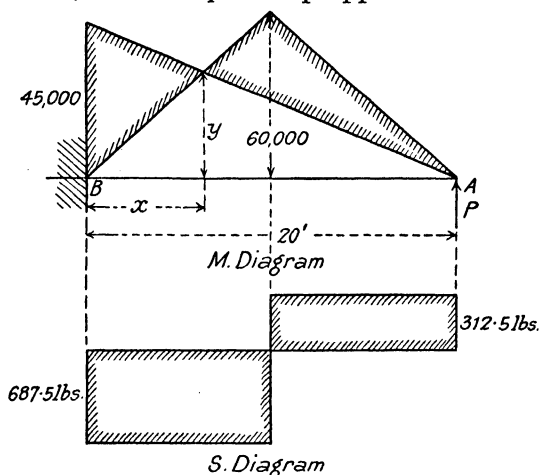



FIG. 81

Then $M_A l + 2M_B(l + l) + M_{A1} l$

$$= 6\left\{\frac{\frac{Wl}{4} \times \frac{l}{2} \times \frac{l}{2}}{l} + \frac{\frac{Wl}{4} \times \frac{l}{2} \times \frac{l}{2}}{l}\right\}$$

Since A and A_1 are supported, $M_A = M_{A1} = 0$

$$\therefore\ 2 \times 2lM_B = \frac{6}{l}\left\{\frac{2Wl^3}{16}\right\}$$

$$M_B = \frac{3}{16}\,Wl$$

$$= \frac{3}{16} \times 1000 \times 240$$

$$= 45{,}000 \text{ lb. in.}$$

$$S_A = \frac{W}{2} + \frac{M_A - M_B}{l} = \frac{1000}{2} + \frac{0 - 45,000}{240} = 500 - 187{\cdot}5$$

$$= 312\ 5 \text{ lb.}$$

$$S_B = -W + S_A = -1000 + 312{\cdot}5$$

$$= -687{\cdot}5 \text{ lb.}$$

To get the distance of the point of contraflexure from B. We have the positive and negative bending moments, equal at this point, and hence

$$\frac{x}{y} = \frac{120}{60,000} = \frac{1}{500}$$

$$\therefore \quad y = 500x \quad . \quad . \quad . \quad . \quad . \quad (1)$$

also

$$\frac{y}{45,000} = \frac{240 - x}{240}$$

$$\therefore \quad y = \frac{45,000(240 - x)}{240}$$

$$= \frac{1500}{8}(240 - x) \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad 500x = \frac{1500}{8}(240 - x)$$

$$x = \frac{3}{8}(240 - x)$$

$$8x = 720 - 3x$$

$$x = 65{\cdot}45 \text{ in.}$$

99. **Advantages and Disadvantages of Continuous Beams.** The maximum bending moment on a continuous beam is less than that which occurs if the spans are bridged by simply supported beams; also the maximum bending moment occurs at the supports. It would appear that a decided advantage is gained by the use of a continuous beam, since lighter material will be required to resist the bending moment, and also, since the heavy portion of the beam, required to resist the bending moment, is near the supports, the weight of the girder will not materially affect the stresses.

It must be remembered, however, that the bending moment is calculated on the assumption that the supports are all at the same level. Any deviation from this condition seriously affects the bending moment, and consequently the estimated stresses. It is owing to the difficulty of arranging all the supports at the same level, and keeping them at this level, that continuous beams are not more universally adopted.

When a continuous beam consists of a number of spans, two points of contraflexure usually occur between each pair of supports. The bending moment being zero at these points, it is possible to hinge the girder here. The end portions then act as cantilevers, and the central portion as a simply supported beam ; hence changes in level of the supports do not affect the bending moment. This is the principle underlying the cantilever bridge, an excellent example being the Forth Bridge.

EXAMPLE 4.

A cantilever is built into a wall and propped level when unloaded. Find where the prop must be placed in order that it shall support one-half of an evenly distributed load that may be placed on the cantilever. (Lond. Univ., 1913.)

Let the constraint be A, the prop B, and the free end C. Then, imagining a similar span to the left of A, we can get the value of M_A.

$M^1_B = M_B = \frac{wl^2}{2}$ where $l = BC$ and w is the evenly distributed load.

Then, if $AB = l_1$, $AC = l_1 + l = L$.

$$M_B l_1 + 2M_A(l_1 + l_1) + M^1_B l_1 = 6\left\{\frac{\frac{wl_1^2}{8} \times \frac{2}{3} l_1 \times \frac{l_1}{2}}{l_1}\right\}2$$

$$wl^2 l_1 + 4M_A l_1 = \frac{wl_1^3}{2}$$

$$wl^2 + 4M_A = \frac{wl_1^2}{2}$$

$$\therefore \qquad 4M_A = \frac{wl_1^2}{2} - wl^2$$

$$M_A = \left\{\frac{wl_1^2}{2} - wl^2\right\}\frac{1}{4}$$

also $$R_B = \frac{wl_1}{2} + \frac{M_B - M_A}{l_1} + \frac{1}{2}wl + \frac{M_B - M_C}{l} = \frac{w(l_1 + l)}{2}$$

$$\therefore \quad \frac{wl_1}{2} + \frac{wl^2}{2l_1} - \frac{wl_1}{8} + \frac{wl^2}{4l_1} + \frac{wl}{2} + \frac{wl}{2} - 0 = \frac{w(l_1 + l)}{2}$$

$$l_1 + \frac{l^2}{l_1} - \frac{l_1}{4} + \frac{l^2}{2l_1} + l + l = l_1 + l$$

$$l_1{}^2 - 4ll_1 - 6l^2 = 0$$

$$\therefore l_1 = 5{\cdot}16l$$

$$= 0{\cdot}840L$$

EXAMPLES IX

1. Deduce a formula for the deflection of a beam of constant section when loaded at the centre and fixed horizontally at the ends. A timber beam fixed horizontally at the ends has a depth of 12 in., and a width of 5 in.; its span is 20 ft. Find: (1) The load which, placed at the centre of the beam, will produce a maximum stress of 1,000 lb./sq. in. on a vertical section of the beam; (2) The deflection under this load. $E = 1{,}750{,}000$ lb./sq. in. (Lond. Univ., 1918.) *Ans.*, 4,000 lb., 0·228 in.

2. A beam with ends fixed horizontally at the same level is subjected to a uniformly distributed load. Obtain expressions for the bending moment, slope, and deflection at any point, and determine their maximum values. (Lond. Univ., 1913.)

3. A rolled steel joist 20 ft. clear span is built into walls at each end, and supports two loads of 5 tons each at a distance of 7 ft. from each wall. Sketch the bending moment and shear diagrams. Find a suitable section for the beam, the depth of which is 3 times the breadth of the flanges and 12 times the thickness of the flanges. Maximum stress, 6 tons/sq. in. (Lond. Univ., 1914.) *Ans.*, 11·25 in. × 3·75 in. × $\frac{15}{16}$ in.

4. A T-shaped mild steel bar is used to span 10 ft. and is rigidly built in at its ends. Loads of 1 ton each are placed at 2, 4, 6, and 8 ft. respectively from one end. Find the maximum tensile and compressive stresses in the bar. Size of bar, 10 in. deep by 6 in. across by 1 in. thick, at all places. (Lond. Univ., 1922.) *Ans.*, 1·11 and 2·0 tons/sq. in. respectively.

5. A cantilever 12 ft. long carries a uniformly distributed load of 250 lb./ft. run. It is supported at a point 8 ft. from the wall by a column, which causes the cantilever to be forced upwards to such an extent that the bending moment at the wall is zero. Find the pressure on the column, and sketch the diagrams of bending moment and shearing force for the cantilever, with the column in the given position, putting in the leading dimensions. (A.M.I.Mech.E., 1924.) *Ans.*, 2,250 lb.

6. Draw the bending moment and shearing force diagrams for a continuous beam of two spans of length 60 and 40 ft. respectively, when the longer span is loaded uniformly with 1·2 tons per foot run, and the shorter span has a central load of 30 tons. What is the reaction at the central support? *Ans.*, 68·2 tons.

7. Construct to scale the bending moment diagram for a continuous beam of three spans of lengths, from left to right, 40 ft., 30 ft., and 40 ft. respectively. The left-hand span is loaded uniformly with 200 lb./ft. The centre span is not loaded and the right-hand span supports a central load of 800 lb.

8. A continuous beam of two spans, of length l_1 and l_2, is clamped at one end and freely supported at the other end. A load W is placed in the span l_1, at a distance a from the freely-supported end. Find an expression for the bending moment at the clamp and at the central support.

CHAPTER X

BEAMS WITH LARGE ORIGINAL CURVATURE

100. THIS problem has been dealt with by Winkler, and later by Andrews and Pearson. For a full discussion of the theories put forward the student is referred to Todhunter and Pearson's *History of Elasticity* (Vol. II), and to Draper's Company *Research Memoirs*, I, 1904. Each theory diverges slightly from the true state of affairs, the former assuming that longitudinal fibres of the bar, parallel to the central axis, exert no action on each other, and the latter in that it neglects stresses in a radial direction.

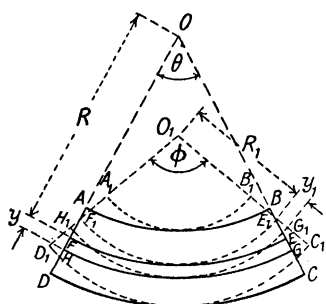



FIG. 82

Making the assumption that a section which is a plane before bending remains a plane after bending, and that the stress is proportional to the strain, we obtain, by consideration of Fig. 82, an idea of the bending moment, and stress due to bending, at any point in the beam. *ABCD* is a portion of the original curved beam in its unstrained state, where R is the radius of curvature of the central axis FE, and y is the distance of a fibre GH from FE. $A_1B_1C_1D_1$ is the strained position of the beam, R_1 being the radius of curvature of F_1E_1, and y_1 being the distance between F_1E_1 and G_1H_1. If f is the stress in the strained fibre G_1H_1, and M the bending moment on the beam, assumed uniform, tending *to increase* curvature, we have, by consideration of the strains in G_1H_1 and in F_1E_1, that

$$f = E\left(\frac{G_1H_1}{GH} - 1\right) = E\left(\frac{R_1 + y_1}{R + y}\frac{\phi}{\theta} - 1\right)$$

$$\therefore \qquad \frac{f}{E} + 1 = \frac{R_1 + y_1}{R + y}\frac{\phi}{\theta} \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if f_c is the stress in the fibre F_1E_1

$$\frac{f_c}{E} + 1 = \frac{R_1}{R}\frac{\phi}{\theta} \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$\therefore \qquad \frac{\frac{f}{E} + 1}{\frac{f_c}{E} + 1} = \frac{R_1 + y_1}{R + y} \times \frac{R}{R_1}$$

$$= \frac{1 + \frac{y_1}{R_1}}{1 + \frac{y}{R}} \quad . \quad . \quad . \quad . \qquad (3)$$

but $\frac{f}{E} = e$ = strain in fibre G_1H_1

and $\frac{f_c}{E} = e_0$ = strain in fibre F_1E_1

$$\therefore \qquad \frac{e + 1}{e_0 + 1} = \frac{1 + \frac{y_1}{R_1}}{1 + \frac{y}{R}} \quad . \quad . \quad . \quad . \qquad (4)$$

which reduces to

$$e = \frac{(1 + e_0)\left(1 + \frac{y_1}{R_1}\right)}{\left(1 + \frac{y}{R}\right)} - 1$$

or, neglecting the difference between y_1 and y

$$e = e_0 + \frac{(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y}{1 + \frac{y}{R}} . \quad . \qquad (5)$$

The stress in G_1H_1 (which is tensile)

$$= Ee$$

$$= E\left(e_0 + \frac{(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y}{1 + \frac{y}{R}}\right) \quad . \qquad (6)$$

The total force on the cross-section $= \int Ee \,.\, dA$

$$= F = \int Ee_0 \,.\, dA + \int \frac{E(1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y}{1 + \frac{y}{R}} \,.\, dA$$

$$= Ee_0A + \int \frac{E(1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y \,.\, dA}{1 + \frac{y}{R}}$$

where A is the cross-sectional area of the beam.

The resisting moment is given by

$$M = \int EeydA$$

$$= \int Ee_0y \,.\, dA + \int \frac{E(1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) y^2dA}{1 + \frac{y}{R}}$$

and since FE passes through the centroid of the section, $Ee_0 \, y \,.\, dA = 0$.

Also if $\int \frac{y^2 \,.\, dA}{1 + \frac{y}{R}}$ be equated to Ah^2

then $$M = E(1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) Ah^2 \quad . \quad . \quad (7)$$

$$\int \frac{y}{1+\frac{y}{R}} dA = \int ydA - \int \frac{y^2}{y+R} dA = -\frac{Ah^2}{R}$$

$\therefore$ $$F = Ee_0A - E(1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{Ah^2}{R}$$

or $$F = EA\left(e_0 - (1+e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{h^2}{R}\right) \quad . \quad (8)$$

From (6) and (7)

$$f = Ee_0 + E(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{y}{1 + \frac{y}{R}}$$

$$= Ee_0 + \frac{M}{Ah^2}\,\frac{yR}{y + R}$$

and, since we have assumed uniform bending moment, $F = 0$.

$$\therefore \qquad e_0 = (1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{h^2}{R}$$

$$= \frac{M}{EAR} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\therefore \qquad f = \frac{M}{AR} + \frac{M}{Ah^2}\,\frac{yR}{y + R}$$

$$= \frac{M}{AR}\left(1 + \frac{y}{y + R}\,\frac{R^2}{h^2}\right) \text{ tensile} \quad . \quad . \quad . \quad (10)$$

On the side of FE next the centre of curvature, y will be negative and the " tensile " stress will be given by

$$f = \frac{M}{AR}\left(1 - \frac{y}{R - y}\,\frac{R^2}{h^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The result will be negative, showing that the stress is really compressive. Hence, we may say that the *compressive* stress on the side next the centre of curvature is given by

$$f = \frac{M}{AR}\left(\frac{y}{R - y}\,\frac{R^2}{h^2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If a bending moment M tends to *decrease* the curvature, then (10) will be a compressive stress, and (12) will be a tensile stress.

101. **Values of h^2.** If the beam section is rectangular of breadth B and depth D, then

$$h^2 = \frac{1}{BD}\int_{-\frac{D}{2}}^{+\frac{D}{2}} \frac{Ry^2\,Bdy}{y + R}$$

$$= \frac{R^3}{D}\log_e \frac{2R + D}{2R - D} - R^2 \quad . \quad . \quad . \quad . \quad (1)$$

If the section is circular, of diameter D, then

$$h^2 = \frac{4}{\pi D^2}\int_{-\frac{D}{2}}^{+\frac{D}{2}} \frac{2Ry^2\left(\sqrt{\frac{D^2}{4} - y^2}\right) dy}{R + y}$$

$$= \frac{D^2}{16} + \frac{1}{2}\left(\frac{D^2}{8R}\right)^2 + \frac{15}{48}\left(\frac{D}{4R}\right)^4 D^2 + \text{etc.} \quad . \quad (2)$$

If the section is a trapezium, whose parallel sides are B and C, and whose centroid is a distance E from B and F from C, then

$$h^2 = \frac{R^3}{A}\left\{\left(C + \frac{B - C}{E + F}(E + R)\right)\log_e \frac{R + E}{R - F} - (B - C)\right\} - R^2 \quad (3)$$

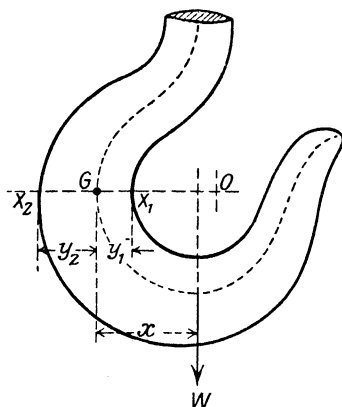



FIG. 83

102. **Application of Formula to Crane Hooks.** When a hook is subjected to a load W, the stress, at the inner and outer side of a horizontal cross-section, such as X_1X_2 (Fig. 83), may be found by calculating the bending stresses at these points by the aid of (10) and (12), par. 100, and adding the direct stress on X_1X_2 due to W.

The bending moment M tending to *reduce* the curvature $= Wx$, and if f_1 is the tensile stress at X_1, and f_2 the compressive stress at X_2, then

$$f_1 = \frac{Wx}{AR}\left(\frac{y_1}{R - y_1}\frac{R^2}{h^2} - 1\right) + \frac{W}{A} \quad . \quad . \quad . \quad . \quad (1)$$

and $$f_2 = \frac{Wx}{AR}\left(1 + \frac{y_2}{y_2 + R}\frac{R^2}{h^2}\right) - \frac{W}{A} \quad . \quad . \quad . \quad . \quad (2)$$

where $OG = R$ is the radius of curvature of the central line at G; the value of h^2 being found from one of the formulae given in par. 101.

EXAMPLE 1.

A crane hook carries a load of $2\frac{1}{2}$ tons, the line of the load being at a horizontal distance of $1\frac{1}{4}$ in. from the inside edge of a horizontal section through the centre of curvature, the centre of curvature being $1\frac{1}{2}$ in. from the same edge. The horizontal section is a trapezium whose parallel sides are $\frac{1}{2}$ in. and 1 in., and perpendicular distance apart $1\frac{1}{4}$ in. Find the greatest tensile and compressive stresses in the hook.

Referring to par. 101 (3), let $B = 1$ in., $C = \frac{1}{2}$ in., then taking moments about B

$$\frac{1 + \frac{1}{2}}{2} \times 1\tfrac{1}{4} \times E = \frac{1}{2} \times \frac{(1\frac{1}{4})^2}{2} + \frac{1}{2} \times \frac{1\frac{1}{4}}{2} \times \frac{1\frac{1}{4}}{3}$$

$$\therefore \qquad E = 0{\cdot}555 \text{ in.}$$

$$h^2 = \frac{R^3}{A}\left\{\left(C + \frac{B - C}{E + F}(E + R)\right)\log_e \frac{E + R}{R - F} - (B - C)\right\} - R^2$$

and $R = 0{\cdot}555 + 1{\cdot}5 = 2{\cdot}055$ in., $A = 0{\cdot}9375$ sq. in., $F = 1{\cdot}25 - 0{\cdot}555 = 0{\cdot}695$ in.

$$\therefore h^2 = \frac{(2{\cdot}055)^3}{0{\cdot}9375}\left\{\left(0{\cdot}5 + \frac{0{\cdot}5}{1{\cdot}25}\,2{\cdot}61\right)2{\cdot}3\log\frac{2{\cdot}61}{1{\cdot}36} - 0{\cdot}5\right\} - (2{\cdot}055)^2$$

$$= 0{\cdot}51$$

From par. 102

$$f_1 = \frac{Wx}{AR}\left(\frac{y_1}{R - y_1}\frac{R^2}{h^2} - 1\right) + \frac{W}{A}$$

where $y_1 = E = 0{\cdot}555$ in. and $x = 1{\cdot}25 + 0{\cdot}555 = 1{\cdot}805$ in.

$$= \frac{2{\cdot}5 \times 1{\cdot}805}{0{\cdot}937 \times 2{\cdot}055}\left(\frac{0{\cdot}555}{1{\cdot}5} \cdot \frac{(2{\cdot}055)^2}{0{\cdot}51} - 1\right) + \frac{2{\cdot}5}{0{\cdot}9375}$$

$$= (2{\cdot}335 \times 2{\cdot}06) + 2{\cdot}67$$

$$= 7{\cdot}47 \text{ tons/sq. in. (tension)}$$

$$f_2 = \frac{Wx}{AR}\left(1 + \frac{y_2}{R + y_2}\frac{R^2}{h^2}\right) - \frac{W}{A}$$

where $y_2 = F = 0{\cdot}695$ in.

$$= 2{\cdot}335\left(1 + \frac{0{\cdot}695}{2{\cdot}055 + 0{\cdot}695}\,\frac{(2{\cdot}055)^2}{0{\cdot}51}\right) - 2{\cdot}67$$

$$= 7{\cdot}2 - 2{\cdot}67$$

$$= 4{\cdot}53 \text{ tons/sq. in. (compression)}$$

103. **Application of Formula to Rings.** The circular ring shown in Fig. 84 has a pull W, applied at each end of a diameter.

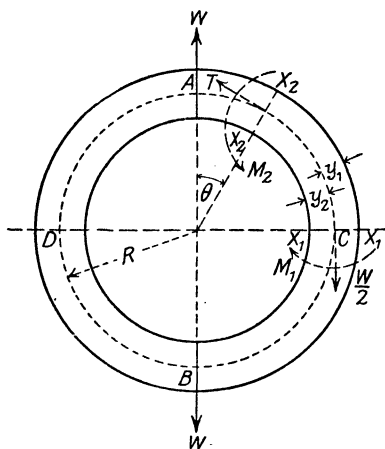



Fig. 84

Considering the portion $X_1X_2X_2X_1$, it is acted on by a couple M_1 at X_1X_1, a pull equal to $\frac{W}{2}$ on the section X_1X_1, a couple M_2 at X_2X_2 and a pull T on the section X_2X_2.

Taking moments about X_2X_2

$$M_2 = M_1 + \frac{WR}{2}(1 - \sin\theta) \quad . \quad . \quad . \quad . \quad (1)$$

and from (7), par. 100

$$E(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2 = M_1 + \frac{WR}{2}(1 - \sin\theta) \quad . \quad (2)$$

or $E(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)Ah^2Rd\theta = M_1Rd\theta + \frac{WR^2}{2}(1 - \sin\theta)\,d\theta$

and integrating for $\theta = 0$ to $\theta = \frac{\pi}{2}$

$$E\int_0^{\frac{\pi}{2}} (1+e_0)\frac{Ah^2R}{R_1}d\theta - E\int_0^{\frac{\pi}{2}}(1+e_0)Ah^2d\theta$$

$$= \int_0^{\frac{\pi}{2}} M_1Rd\theta + \int_0^{\frac{\pi}{2}}\frac{WR^2}{2}(1-\sin\theta)d\theta$$

and since $\displaystyle\int_0^{\frac{\pi}{2}}\frac{(1+e_0)R}{R_1}d\theta = \frac{\pi}{2}$

$$\therefore \quad EAh^2\frac{\pi}{2} - EAh^2\frac{\pi}{2}(1+e_0) = M_1R\frac{\pi}{2} + \frac{WR^2}{2}\left(\frac{\pi}{2}-1\right)$$

or $$-EAh^2\frac{\pi}{2}e_0 = M_1R\frac{\pi}{2} + \frac{WR^2}{2}\left(\frac{\pi}{2}-1\right) \quad . \quad . \quad . \quad (3)$$

Also $$T = \frac{W}{2}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or by (8), par. 100

$$EA\left(e_0 - (1+e_0)\left(\frac{1}{R_1}-\frac{1}{R}\right)\frac{h^2}{R}\right) = \frac{W}{2}\sin\theta$$

from which with (2), above, divided by R gives

$$e_0 = \left(\frac{W}{2}+\frac{M_1}{R}\right)\frac{1}{EA} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$\therefore$ (3) becomes

$$-h^2\left(\frac{W}{2}+\frac{M_1}{R}\right) = M_1R + \frac{WR^2}{2} - \frac{WR^2}{\pi}$$

$$M_1 = \frac{WR}{2}\left(\frac{R^2}{R^2+h^2}\cdot\frac{2}{\pi}-1\right) \quad . \quad . \quad . \quad . \quad (6)$$

hence $$M_2 = \frac{WR}{2}\left(\frac{R^2}{R^2+h^2}\,\frac{2}{\pi}-\sin\theta\right) . \quad . \quad . \quad . \quad (7)$$

M_2 will therefore be maximum when $\theta = 0$ and $180°$

$$M_{max} = \frac{WR}{2}\,\frac{R^2}{R^2+h^2}\,\frac{2}{\pi}$$

$$= \frac{WR^3}{\pi(R^2+h^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From (7), $M_2 = 0$ when $\sin\theta = \dfrac{2R^2}{\pi(R^2 + h^2)}$

Thus, four points will be found (one in each quadrant), at each of which the bending moment, and consequently the bending stress, will be zero.

From (5) and (6)

$$e_0 = \frac{W}{EA\pi} \cdot \frac{R^2}{R^2 + h^2} \cdot \qquad (9)$$

and the stress is given by Ee

$$= E\left(e_0 + \frac{(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)y}{1 + \frac{y}{R}}\right) \text{ by (6) par. 100}$$

and from (2) and (7),

$$E(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right)\frac{y}{1 + \frac{y}{R}} = \frac{WR}{2Ah^2}\left(\frac{R^2}{R^2 + h^2}\frac{2}{\pi} - \sin\theta\right)\frac{yR}{y + R}$$

$$\therefore f = \frac{W}{A}\left[\frac{R^2}{\pi(R^2 + h^2)} + \frac{R^2}{2h^2}\left(\frac{R^2}{R^2 + h^2} \cdot \frac{2}{\pi} - \sin\theta\right)\frac{y}{y + R}\right] \quad (10)$$

The resultant stress at any point in a section will be the sum of the bending stress and the direct stress, and at X_2X_2 the resultant stress is given by

$$f_r = \frac{W}{A}\left[\frac{R^2}{\pi(R^2 + h^2)} + \frac{R^2}{2h^2}\left(\frac{R^2}{R^2 + h^2} \cdot \frac{2}{\pi} - \sin\theta\right)\frac{y}{y + R}\right] + \frac{W\sin\theta}{2A} \quad . \quad . \quad (11)$$

On a section taken along the line of action of W, the stresses are as follows—

(*a*) At *outside* of ring

$$f_r = \frac{W}{\pi A} \cdot \frac{R^2}{R^2 + h^2}\left[1 + \frac{R^2}{h^2} \cdot \frac{y_1}{y_1 + R}\right] \text{(tensile)} \quad . \quad . \quad (12)$$

(*b*) At *inside* of ring

$$f_r = \frac{W}{\pi A} \cdot \frac{R^2}{R^2 + h^2} \left[\frac{R^2}{h^2} \frac{y_2}{R - y_2} - 1 \right] \text{ (compression)} \quad . \quad . \quad (13)$$

and is the *greatest* stress in the ring.

On a section along CD, perpendicular to the line of action of W, the stresses are given by—

(*a*) At *outside* of ring

$$f_r = \frac{W}{A} \left[\frac{R^2}{\pi(R^2 + h^2)} + \frac{R^2}{2h^2} \left(\frac{R^2}{R^2 + h^2} \cdot \frac{2}{\pi} - 1 \right) \frac{y_1}{y_1 + R} \right]$$

$$- \frac{W}{2A} \text{ (compression)} \quad . \quad . \quad . \quad . \quad (14)$$

(*b*) At *inside* of ring

$$f_r = \frac{W}{A} \left[\frac{R^2}{2h^2} \left(\frac{R^2}{R^2 + h^2} \cdot \frac{2}{\pi} - 1 \right) \frac{y_2}{R - y_2} - \frac{R^2}{\pi(R^2 + h^2)} \right]$$

$$+ \frac{W}{2A} \text{ (tension)} \quad . \quad . \quad . \quad . \quad (15)$$

104. **Application of Formula to Links.** Referring to Fig. 85, if M_1 is the bending moment at X_1X_1, and M_2 the bending moment at X_2X_2, then

$$M_2 = M_1 + \frac{WR}{2}(1 - \sin\theta) \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and } E(1 + e_0)\left(\frac{1}{R_1} - \frac{1}{R}\right) Ah^2 = M_1 + \frac{WR}{2}(1 - \sin\theta) \quad . \quad (2)$$

or as in par. 103

$$E\int_o^{\frac{\pi}{2}} (1 + e_0)Ah^2 \frac{Rd\theta}{R_1} - E\int_o^{\frac{\pi}{2}} (1 + e_0)Ah^2 \frac{Rd\theta}{R}$$

$$= \int_o^{\frac{\pi}{2}} M_1 R d\theta + \int_o^{\frac{\pi}{2}} \frac{WR^2}{2}(1 - \sin\theta)d\theta$$

but

$$\int_o^{\frac{\pi}{2}} (1 + e_0) \frac{Rd\theta}{R_1} = \frac{\pi}{2} - \frac{M_1 a}{2EI}$$

$$\therefore \quad EAh^2\frac{\pi}{2} - EAh^2\frac{M_1 a}{2EI} - EAh^2(1+e_0)\frac{\pi}{2}$$

$$= M_1 R\frac{\pi}{2} + \frac{WR^2}{2}\left(\frac{\pi}{2} - 1\right)$$

or $$M_1\left(\frac{\pi R}{2} + \frac{EAh^2 a}{2EI}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - EAh^2 e_0 \frac{\pi}{2}$$

and from (5), par. 103, $e_0 = \left(\frac{W}{2} + \frac{M_1}{R}\right)\frac{1}{EA}$

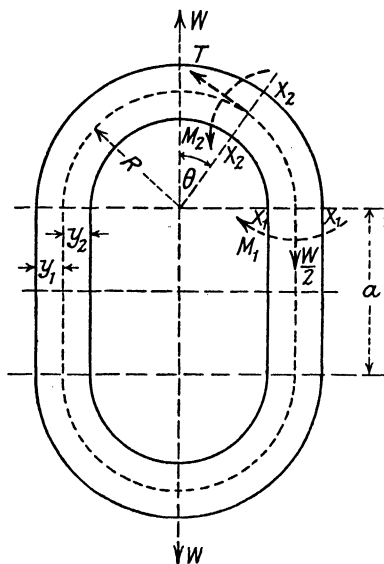



Fig. 85

$$\therefore \quad M_1\left(\frac{\pi R}{2} + \frac{EAh^2 a}{2EI}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - h^2\frac{\pi}{2}\frac{W}{2} - \frac{M_1 h^2 \pi}{2R}$$

$$M_1 = \frac{\dfrac{WR^2}{2}\left(1 - \dfrac{\pi}{2}\right) - \dfrac{W\pi h^2}{4}}{\dfrac{\pi R}{2} + \dfrac{h^2 a}{2k^2} + \dfrac{h^2 \pi}{2R}}$$

$$= \frac{W\left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}\right)}{R + \frac{h^2 a}{\pi k^2} + \frac{h^2}{R}} \quad . \quad . \quad . \quad . \quad (3)$$

$$M_2 = \frac{W\left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}\right)}{R + \frac{h^2 a}{\pi k^2} + \frac{h^2}{R}} + \frac{WR}{2}(1 - \sin\theta) \quad . \quad (4)$$

where k is the radius of gyration of the section.

In a numerical example, having found the value of M_1, the value of e_0 can then be calculated, from which the stress $= Ee$ is obtained.

The stress at the inside or outside of the link, at any section, can then be ascertained, as in (12)–(15), par. 103. The greatest bending stress will be that at the inside of the link when $\theta = 0$.

EXAMPLE 2.

A link is subjected to a pull of 2 tons. It is composed of steel, 1 in. diameter, and has a mean radius of $1\frac{1}{4}$ in. Its semicircular ends are connected by straight pieces 1 in. long. Estimate the maximum compressive stress in the link and the tensile stress at the same section.

$$h^2 \doteq \frac{D^2}{16} + \frac{1}{2}\left(\frac{D^2}{8R}\right)^2 = \frac{1}{16} + \frac{1}{2}\left(\frac{1}{10}\right)^2$$

$$= 0{\cdot}0675.$$

$$M_1 = \frac{W\left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{h^2}{2}\right)}{R + \frac{h^2 a}{\pi k^2} + \frac{h^2}{R^2}}$$

$$= \frac{2\left(\frac{1{\cdot}25^2}{\pi} - \frac{1{\cdot}25^2}{2} - \frac{0{\cdot}0675}{2}\right)}{1{\cdot}25 + \frac{0{\cdot}0675}{\pi} \times 16 + \frac{0{\cdot}0675}{1{\cdot}56}}$$

$$= -\,0{\cdot}388 \text{ ton in.}$$

$$e_0 = \left(\frac{W}{2} + \frac{M_1}{R}\right)\frac{1}{EA} = \left(1 - \frac{0{\cdot}388}{1{\cdot}25}\right)\frac{1}{0{\cdot}7854E}$$

$$= \frac{0{\cdot}878}{E}$$

$$f = Ee_0 + \frac{WR}{2Ah^2}\left(\frac{R^2}{R^2+h^2}\frac{2}{\pi} - \sin\theta\right)\frac{yR}{y+R} + \frac{W\sin\theta}{2A}$$

Maximum compressive stress occurs when $\theta = 0$ and $y = -0{\cdot}5$.

$$\therefore f_{max} = 0{\cdot}878 + \frac{2 \times 1{\cdot}25}{2 \times 0{\cdot}7854 \times 0{\cdot}0675}\left(\frac{1{\cdot}56}{1{\cdot}6275} \times \frac{2}{\pi}\right) \times -\frac{0{\cdot}5 \times 1{\cdot}25}{0{\cdot}75}$$

$$= 0{\cdot}878 - 12{\cdot}04$$

$$= -11{\cdot}162$$

$= 11{\cdot}162$ tons/sq. in. compression

$$f_{tensile} = 0{\cdot}878 + \frac{2 \times 1{\cdot}25}{2 \times 0{\cdot}7854 \times 0{\cdot}0675}\left(\frac{1{\cdot}56}{1{\cdot}6275} \times \frac{2}{\pi}\right)\frac{0{\cdot}5 \times 1{\cdot}25}{1{\cdot}75}$$

$$= 0{\cdot}878 + 5{\cdot}16$$

$= 6{\cdot}038$ tons/sq. in.

105. **Flat Spiral Springs.** A spring of this type is shown by Fig. 86. One end is fastened at O, and the other is attached to a spindle. Suppose a couple of magnitude M to be applied to the spindle, then there will be a reaction F at O, whose horizontal and vertical components are F_x and F_y respectively.

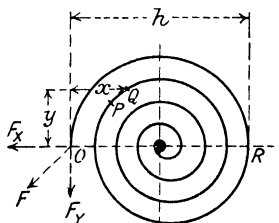



FIG. 86

Considering the small portion PQ, the bending moment is equal to $F_y x - F_x y$.

Then, if $d\theta$ is the angle, through which tangents at the ends of PQ change, we have

$$\text{Change of curvature} = \frac{d\theta}{ds} \text{ where } ds = PQ.$$

$$\text{But change of curvature} = \frac{\text{bending moment}}{EI}$$

$$= \frac{F_Y x - F_X y}{EI}$$

$$\therefore \qquad d\theta = \frac{F_Y x - F_X y}{EI} ds$$

and $$\theta = \frac{1}{EI}\left[F_Y \int x ds - F_X \int y ds\right]$$

= total angle turned through in " winding-up,"

the last term of which is zero.

Also $\int x ds \doteqdot L \frac{h}{2}$ where L is the length of the spring and $F_Y \frac{h}{2} = M$, the couple applied to the spindle.

$$\therefore \qquad \theta = \frac{ML}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The energy stored in the spring

$$= \tfrac{1}{2} M\theta$$

$$= \tfrac{1}{2} \frac{M^2 L}{EI} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the coils have a width b, and depth d, and f_1 is the stress at PQ, then

$$f_1 = \frac{M_1 \frac{d}{2}}{I}$$

where M_1 = the bending moment at PQ.

The maximum stress occurs at R, and is given by

$$f_{max} = \frac{F_Y h \times 6}{bd^2}$$

$$= \frac{12M}{bd^2} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the greatest allowable stress is f_2, then the greatest allowable value of

$$M = \frac{bd^2 f_2}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and hence, from (2), the maximum resilience

$$= \frac{1}{2}\left(\frac{bd^2f_2}{12}\right)^2 \frac{L \times 12}{Ebd^3}$$

$$= \frac{1}{24}\frac{f_2^2}{E} \text{ per unit volume} \quad . \quad . \quad . \quad (5)$$

EXAMPLES X

1. A beam of rectangular section 3 in. × 4 in. has its central line curved to a radius of 6 in. The beam is subjected to a bending moment of 12 ton in. Find the greatest tension and compression stresses in the beam if the bending moment tends to increase the curvature.

Ans., 1·3 and 2·1 tons/sq. in. respectively.

2. Prove that the moment of resistance M of a curved beam of initial radius R_0, when bent to a radius R_1 by a uniform bending moment, may be expressed by

$$M = EAh^2\left(\frac{1}{R_1} - \frac{1}{R_0}\right) \text{ where } Ah^2 = \sum \frac{R_0 y^2}{R_0 + y}$$

and y is the distance of any point in the cross-section A from the plane passing through the centre of figure and perpendicular to the plane of bending. (Lond. Univ., 1913.)

3. A circular ring is made of ¾-in. diameter iron, the mean radius of the ring being 2 in. The ring is employed at the end of a chain for lifting a half-ton load; find the maximum stress in the ring and show where it occurs. Deduce any formula you use. (Lond. Univ., 1913.)

Ans., 19,750 lb./sq. in. compression under load at inside of ring.

4. The section of a crane hook is a trapezium whose inner and outer sides are 2 in. and 1 in. respectively and depth 2½ in. The centre of curvature of the section is at a distance of 2½ in. from the inside of the section, and the load line is 2 in. from the same point. Find the greatest load the hook will carry if the maximum stress is not to exceed 7 tons/sq. in.

Ans., approx. 10 tons.

5. A chain link is made of ½ in. round steel and is semicircular at each end, the mean diameter of which is 1¼ in. The straight sides of the link are each ¾ in. long. If the link carries a load of ½ ton, estimate the greatest tensile and compressive stresses in the link.

Ans., 6·375 and 11·8 tons/sq. in. respectively.

6. A flat spiral spring has a rectangular section ½ in. × $\frac{1}{50}$ in., and a length of 12 ft. If the maximum stress in the spring is 90,000 lb./sq. in., find the couple exerted on its central spindle, and the number of turns the spindle can make before the spring is run down. What amount of energy is stored in the spring? $E = 30 \times 10^6$ lb./sq. in. *Ans.*, 1·5 lb. in., 3·44, 16·2 in. lb.

CHAPTER XI

THE TORSION OF SHAFTS

106. The Relation between Stress, Strain, and Angle of Twist. A cylindrical rod is said to be subject to pure torsion when the torsion is caused by a couple, applied so that the axis of the couple coincides with the axis of the rod. The state of stress, at any point in the cross-section of the rod, is one of pure

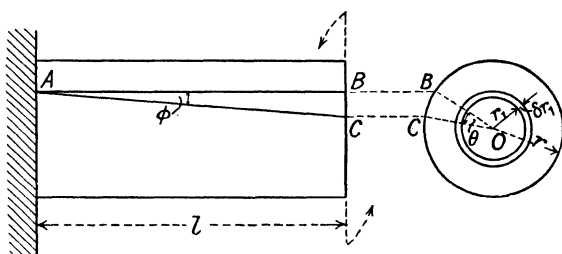



FIG. 87

shear, and the strain is such that one cross-section of the rod moves relative to another.

Considering the cylindrical rod of length l and radius r, shown in Fig. 87, a couple of magnitude T is applied to one end, and the other end of the rod is held, or constrained, by a balancing couple of equal magnitude. A line AB, on the surface of the rod, which is parallel to the axis before strain, takes up the form of a long helix AC after strain; the angle ϕ being the shear strain of the material at the surface, and since this angle is small

$$BC = l\phi$$

or

$$\phi = \frac{BC}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

but $\phi = \frac{f_s}{C}$ where f_s is the shear stress in the material at the surface of the rod, and $B\hat{O}C$ is the angular movement of a radius OB due to strain in the length l.

$$\therefore \quad f_s = \phi C \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or $$f_s = \frac{r\theta}{l} C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which may also be put in the form

$$f_s = kr \text{ where } k = \frac{\theta}{l} C$$

hence, if f_s^1 is the shear stress at a radius r_1, we have

$$\frac{f_s^1}{r_1} = \frac{f_s}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

107. **Relation between Twisting Couple and Shear Stress.** Considering a thin ring in the cross-section at A, of radius r_1, and thickness δr_1, on which the shear stress is f_s^1, the total force on the ring

$$= f_s^1 \times 2\pi r_1 \delta r_1$$

and the moment of this, about the axis of the rod,

$$= f_s^1 \times 2\pi r_1^2 \delta r_1$$

$$= \frac{f_s}{r} \times 2\pi r_1^3 dr_1$$

$$= \frac{f_s}{r} \times 2\pi r_1^3 dr_1$$

when the ring is infinitely thin.

The total resisting moment of the section

$$= 2\pi \frac{f_s}{r} \int_o^r r_1^3 dr_1$$

$$= f_s \frac{\pi r^3}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

but the total resisting moment of the section is equal to the applied couple T

$$\therefore \qquad T = f_s \frac{\pi r^3}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Multiplying the right-hand side above and below by r

$$T = \frac{f_s}{r} \frac{\pi r^4}{2}$$

and $\frac{\pi r^4}{2}$ is the polar moment of inertia of the section I_p.

$$\therefore \qquad T = \frac{f_s}{r} I_p \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and from (3) and (7) we get

$$\left[\frac{T}{I_p} = \frac{f_s}{r} = \frac{C\theta}{l}\right] \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is usually called the torsion equation. The usual units of measurement for the various quantities are—

T = Twisting moment in lb. in.

I_p = Polar moment of inertia of section in inch units.

f_s = Shear stress in lb./sq. in. at a radius r in.

C = Rigidity modulus in lb./sq. in.

θ = Angle of twist in radians in a length of l in.

EXAMPLE 1.

Calculate the size of a shaft, which will transmit 50 h.p. at 110 r.p.m. The shearing stress to be limited to 3 tons per square inch, and the twist of the shaft is not to exceed 1 degree in 7½ ft. of length of shaft. The modulus of rigidity, C, is 5,000 tons per square inch. Assume the twisting moment to be uniform. (Lond. Univ., 1919.)

$$T = 63{,}000 \, \frac{\text{h.p.}}{N}$$

$$= \frac{63{,}000 \times 50}{110} \text{ lb. in.}$$

$$\text{Resisting moment} = f_s \frac{\pi r^3}{2} = \text{torque}$$

$$\therefore \qquad r^3 = \frac{63{,}000 \times 50}{110} \times \frac{2}{3 \times 2240 \times \pi}$$

and $r = 1{\cdot}394$ in. for safe stress.

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\therefore \qquad r^4 = \frac{2Tl}{\pi C\theta}$$

$$= \frac{2 \times 63{,}000 \times 50 \times 90 \times 57{\cdot}3}{110 \times \pi \times 5000 \times 2240 \times 1}$$

$\therefore \qquad r = 1{\cdot}7$ for safe twist.

$\therefore$ Dia. of shaft $= 3{\cdot}4$ in.

$\doteqdot 3\tfrac{7}{16}$ in.

108. **Hollow Circular Shafts.** In equation (3), par. 106, we proved that $f_s = kr$; thus, in a solid shaft of large diameter, a great deal of the material towards the axis will carry very little stress, and consequently will offer very little resistance to the applied couple. In a hollow shaft the average intensity of stress will be greater than that for a solid shaft of the same diameter, and hence a greater resistance to the applied couple will be exerted.

Let $\qquad R =$ outer radius of a hollow shaft
and let $\qquad r =$ inner ,, ,, ,, ,,
also, if $\qquad f_s =$ shear stress at radius R, and
$\qquad f_s^1 =$,, ,, ,, r

then the value of the resisting moment, which is equal to the applied couple T, is given by

$$T = f_s \frac{\pi R^3}{2} - f_s^1 \frac{\pi r^3}{2}$$

$$= \frac{f_s \pi}{2}\left(R^3 - \frac{r^4}{R}\right)$$

$$= \frac{f_s \pi}{2}\left(\frac{R^4 - r^4}{R}\right) \quad . \quad . \quad . \quad . \quad (1)$$

the value of I_p in the twisting equation being given by

$$I_p = \frac{\pi}{2}(R^4 - r^4) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Comparing the strength of a hollow shaft with that of a solid shaft of the same material, weight, and length.

Let R and r be the outer and inner radii of the hollow shaft, and f_s the maximum shear stress; also let R_1 be the radius of the solid shaft.

$$T_{hollow} = \frac{f_s \pi}{2}\left(\frac{R^4 - r^4}{R}\right)$$

$$T_{solid} = \frac{f_s \pi}{2} R_1^3$$

If the maximum stress is the same for each shaft

$$\frac{T_{hollow}}{T_{solid}} = \frac{R^4 - r^4}{RR_1^3}$$

$$= \frac{R^2 + r^2}{RR_1} \text{ since } R_1^2 = R^2 - r^2$$

$$= \frac{R}{R_1}\left(1 + \frac{1}{n^2}\right) \text{ where } r = \frac{R}{n}$$

$$= \frac{n^2 + 1}{n\sqrt{n^2 - 1}} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is very common practice to take $n = 2$, which gives

$$\frac{T_{hollow}}{T_{solid}} = \frac{5}{2\sqrt{3}}$$

$$= 1{\cdot}44$$

109. **Shaft Diameter for a Given Horse-power.**

If T = the mean twisting moment in lb. in.

N = the r.p.m. of the shaft

h.p. = the horse-power to be transmitted

then

$$\text{h.p.} = \frac{T \times 2\pi N}{12 \times 33{,}000} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \quad T = \frac{12 \times 33{,}000}{2\pi}\,\frac{\text{h.p.}}{N}$$

or

$$T = 63{,}000\,\frac{\text{h.p.}}{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the twisting moment is variable, then the maximum twisting moment T_{max} will be equal to a constant multiplied by the value in (2), and from (6), par. 107

$$f_s\,\frac{\pi r^3}{2} = 63{,}000\,\frac{\text{h.p.}}{N}$$

$$\therefore \quad d = 2\sqrt[3]{\frac{2 \times 63{,}000}{\pi f_s}}\,.\,\sqrt[3]{\frac{\text{h.p.}}{N}}$$

or

$$d = k\sqrt[3]{\frac{\text{h.p.}}{N}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where d is the diameter of the shaft and f_s the maximum shear stress.

From equation (3), we see that, for a given value of horse-power, the diameter of the shaft will be smaller for a high speed than that required for a low speed. As an example, we may take the case of a De Laval steam turbine, which develops 15 h.p. on a $\frac{3}{8}$ in. diameter shaft, at 33,000 r.p.m. The diameter of shaft for the same horse-power, at 100 r.p.m., is given by

$$\frac{d^3}{(\frac{3}{8})^3} = \frac{33{,}000}{100}$$

or

$$d = 2 \cdot 59 \text{ in.}$$

EXAMPLE 2.

The external and internal diameters of a hollow steel shaft are 15 in. and 9 in. respectively. Determine what horse-power it will transmit when the speed is 90 r.p.m. The maximum intensity of shear stress is not to exceed 8,000 lb. per square inch. Compare the strength of this shaft with a solid one of the same material and weight. (A.M.I.Mech.E., 1918.)

$$T = \frac{f_s \pi}{2}\left(\frac{R^4 - r^4}{R}\right)$$

$$= \frac{8000\pi}{2}\left(\frac{7 \cdot 5^4 - 4 \cdot 5^4}{7 \cdot 5}\right)$$

$$= \frac{4000\pi \times 2755}{7 \cdot 5} \text{ lb. in.}$$

$$\text{h.p.} = \frac{T \, . \, N}{63{,}000}$$

$$= \frac{4000\pi \times 2755 \times 90}{7 \cdot 5 \times 63{,}000}$$

$$= 6595$$

$$\frac{\text{Strength of hollow shaft}}{\text{Strength of solid shaft}} = \frac{n^2 + 1}{n\sqrt{n^2 - 1}} \text{ where } n = \frac{R}{r}$$

$$= \frac{15^2 + 9^2}{9^2 \times \frac{15}{9}\sqrt{\frac{15^2 - 9^2}{9^2}}}$$

$$= \frac{306}{15 \times 12}$$

$$= 1 \cdot 7$$

110. **Horse-power of Turbine-driven Ships.** In order to determine the power developed by the turbine, we make use of instruments called torsion-meters, mounted on the shaft. A description of these instruments does not enter the province of this work, but the student is referred to *The Theory of Machines*, by R. F. McKay, for a description of several well-known types. For our purpose it is sufficient to know that the torsion-meter is able to measure the angle of twist θ in a given length l of the shaft, while rotation of the shaft is taking place.

From (1), par. 109

$$\text{h.p.} = \frac{T \times 2\pi N}{12 \times 33{,}000}$$

$$= \frac{2\pi}{12 \times 33{,}000} \frac{C\theta}{l} I_p N$$

and for a given shaft C, I_p and l are constant

$$\therefore \qquad \text{h.p.} = x\theta \,.\, N \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus, if x is known and θ and N obtained, we can calculate the power being transmitted along the propeller shaft.

The value of C for the shaft can be found by mounting the shaft in a lathe, locking one end, and applying various torques to the other end, which is free. If the torsion meter be mounted on the shaft, the angle of twist for each torque can be read off. Torque and angle of twist are then plotted, and taking a reading from the graph, and substituting in $\frac{T}{I_p} = \frac{C\theta}{l}$, we get the value of C and hence x.

EXAMPLE 3.

A propeller shaft, diameter $10\frac{1}{2}$ in., is to be subjected to a torque for the calibration of a torsion-meter. Two long pointers are fixed to the shaft at a distance of 8 ft. apart. Arrangements are made to apply a torque which is to be 30 per cent in excess of the twisting moment due to 12,500 h.p. at 500 r.p.m. The movements of the ends of the pointers are observed by means of verniers reading to 0·01 in. What length of pointer is necessary if, at maximum test torque, the margin of error due to vernier reading is limited to 2 per cent. Assume a provisional value for the torsion modulus of the shaft to be 12 millions in pound-inch units. (Lond. Univ., 1921.)

Applied torque

$$= \frac{63{,}000 \times 12{,}500}{500} \times \frac{130}{100}$$

$$= 2{,}047{,}500 \text{ lb. in.}$$

Angle of twist due to this torque

$$= \frac{Tl}{I_p C}$$

$$= \frac{2{,}047{,}500 \times 96 \times 2}{\pi \times (5{\cdot}25)^4 \times 12 \times 10^6} \text{ radians}$$

$$= 0{\cdot}01372 \text{ radians}$$

Let l in. = length of pointers. Correct movement of verniers = $0{\cdot}01372 \times l$ in.

$$\therefore \quad \frac{0{\cdot}01}{0{\cdot}01372\, l} = \frac{2}{100}$$

$$\therefore \quad l = 36{\cdot}44 \text{ in.}$$

111. **Torsional Resilience of Circular Shafts.** If a solid shaft be subjected to a torque which increases gradually from zero to a value T, and θ is the resultant angle of twist, then the energy stored in the shaft, or the resilience of the shaft

$$= \tfrac{1}{2}T\theta$$

$$= \frac{1}{2}\frac{f_s I_p}{r} \times \frac{f_s}{r}\frac{l}{C}$$

$$= \frac{\frac{1}{2}f_s^2}{C}\frac{I_p}{r^2}l$$

$$= \frac{1}{2}\frac{f_s^2}{C}\frac{\pi r^2 l}{2}$$

$$= \frac{1}{4}\frac{f_s^2}{C} \,.\, \text{volume of shaft} \quad . \quad . \quad . \quad (1)$$

In the case of a hollow shaft of radii R and r, we have

$$\text{Resilience} = \tfrac{1}{2}T\theta$$

$$= \frac{1}{2}\frac{T^2}{I_p}\frac{l}{C}$$

$$= \frac{\frac{1}{2}\left\{\frac{\pi f_s}{2}\left(\frac{R^4 - r^4}{R}\right)\right\}^2}{\frac{\pi(R^4 - r^4)}{2}\, C}$$

$$= \frac{1}{4}\frac{f_s^2}{C}\frac{R^2 + r^2}{R^2}\pi(R^2 - r^2)\,l$$

$$= \frac{1}{4}\frac{f_s^2}{C} \cdot \frac{R^2 + r^2}{R^2} \text{ volume of shaft} \qquad . \qquad (2)$$

112. **Torsion of a Taper Shaft.** Suppose a twisting moment T applied to the taper shaft of length L, shown by Fig. 88. The resisting moment of all cross-sections of the shaft must be equal to T, then

Let f_{s1} = maximum shear stress at end of radius r_1

f_{s2} = ,, ,, ,, ,, ,, r_2

f_s = ,, ,, ,, ,, cross-section of radius r

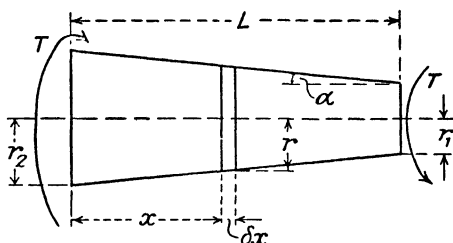



FIG. 88

$$\therefore \qquad \frac{\pi f_{s1} r_1^3}{2} = \frac{\pi f_{s2} r_2^3}{2} = \frac{\pi f_s r^3}{2}$$

or $$f_{s1} r_1^3 = f_{s2} r_2^3 = f_s r^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a parallel shaft we have, by par. 106 (8),

$$\theta = \frac{Tl}{CI_p}$$

Let δx be a small length of the shaft at distance x from the larger end, and whose radius we may assume to be r, if $\delta\theta$ is the angle of twist of the small length δx we have

$$\delta\theta = \frac{T}{CI_p}\,\delta x$$

$$= \frac{2T}{C\pi}\frac{\delta x}{r^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

but $r = r_2 - x \tan a$

$$= r_2 - x \frac{r_2 - r_1}{L}$$

$$= r_2 - ax \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $a = \frac{r_2 - r_1}{L}$

Substituting (3) in (2)

$$\delta\theta = \frac{2T}{C\pi}(r_2 - ax)^{-4}\,\delta x$$

or

$$d\theta = \frac{2T}{C\pi}(r_2 - ax)^{-4}\,dx$$

The total angle of twist θ for the length L is given by

$$\theta = \int d\theta = \int_o^L \frac{2T}{C\pi}(r_2 - ax)^{-4}\,dx$$

$$= \frac{2T}{C\pi}\int_o^L (r_2 - ax)^{-4}\,dx$$

$$= \frac{2T}{C\pi}\frac{1}{3a}\Big|\,(r_2 - ax)^{-3}\,\Big|_o^L$$

$$= \frac{2T}{C\pi}\frac{L}{3(r_2 - r_1)}\left\{\left(r_2 - \frac{(r_2 - r_1)\,L}{L}\right)^{-3} - (r_2)^{-3}\right\}$$

$$= \frac{2T}{C\pi}\frac{L}{3(r_2 - r_1)}\left\{\frac{1}{r_1^3} - \frac{1}{r_2^3}\right\}$$

$$\therefore \quad \theta = \frac{2TL}{C\pi} \times \frac{(r_1^2 + r_1 r_2 + r_2^2)}{3r_1^3 r_2^3} \quad . \quad . \quad (4)$$

In the special case when $r_1 = r_2$, i.e. the shaft is parallel, then

$$\theta = \frac{2T}{C\pi} \times \frac{L}{r_1^4}$$

$$= \frac{TL}{CI_p}$$

which is the result already obtained in par. 106.

113. Torsion of Shafts of Rectangular and Square Section. In estimating the twisting moment on a shaft whose section is not circular, many difficulties arise which make the problem very complex. The student is referred to Todhunter and Pearson's *History of the Theory of Elasticity* for an account of St. Venant's investigation of the problem, in which it is shown that

(*a*) For a *rectangular* section of short side x and long side y

$$T = \frac{x^2y^2}{3y + 1{\cdot}8x} f_s \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and that the greatest value of f_s occurs at the middle point of y.

(*b*) For a *square* section, writing $y = x$ in (1),

$$T = \frac{x^4}{4{\cdot}8x} f_s$$

$$= 0{\cdot}208x^3 f_s \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the maximum value of f_s occurring at the middle point of a side and for rectangular and square sections

$$\theta = \frac{Tl}{C} \cdot \frac{42I_p}{A^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

when $\frac{y}{x}$ is less than 3.

Applying the method of par. 111, we get the resilience of (*a*)

$$= \frac{1}{2} \frac{x^2y^2}{3y + 1{\cdot}8x} f_s \times \frac{Tl}{C} \frac{42I_p}{(xy)^4}$$

$$= \frac{1}{2} \frac{f_s^2}{C} \text{ volume} \times \frac{42(x^2 + y^2)}{12(3y + 1{\cdot}8x)^2} \quad . \quad (4)$$

and hence the resilience of (*b*)

$$= 0{\cdot}156 \frac{f_s^2}{C} \text{ volume} \quad . \quad . \quad . \quad . \quad (5)$$

114. **Torsion Combined with Bending.** This case is exceedingly common in practice. A shaft transmitting power is usually subjected to bending stresses due to its own weight, as in the case of a propeller shaft, or due to the weight of pulleys and the pull of belts as in line shafting. In general, the stresses to be considered are (1) those due to bending,

(2) those due to torsion, (3) those due to shear caused by bending. In the majority of cases (3) is unimportant.

The maximum stress due to torsion is given by

$$f_s = \frac{2T}{\pi r^3} \quad (6) \text{ par. } 106$$

and the maximum stress due to bending is given by

$$f = \frac{4M}{\pi r^3} \quad (5) \text{ par. } 67.$$

The principal stresses are given by

$$f_n = \tfrac{1}{2}\{f \pm \sqrt{f^2 + 4f_s^2}\} \quad (1) \text{ par. } 25.$$

The maximum value being tensile at a point where the bending stress is tensile, and compressive where the bending stress is compressive.

Substituting the values of f_s and f, we get

$$f_n = \frac{1}{2}\left\{\frac{4M}{\pi r^3} + \sqrt{\frac{16M^2}{(\pi r^3)^2} + \frac{16T^2}{(\pi r^3)^2}}\right\}$$

which may be written in either of the following forms—

$$f_n = \frac{2}{\pi r^3}\{M + \sqrt{M^2 + T^2}\} \quad . \quad . \quad . \quad (1)$$

or

$$f_n = \frac{4}{\pi r^3}\{\tfrac{1}{2}(M + \sqrt{M^2 + T^2})\} \quad . \quad . \quad (2)$$

from (1) $$T_\varepsilon = \frac{f_n \pi r^3}{2} = M + \sqrt{M^2 + T^2} \quad . \quad . \quad (3)$$

and from (2) $$M_\varepsilon = \frac{f_n \pi r^3}{4} = \tfrac{1}{2}(M + \sqrt{M^2 + T^2}) \quad . \quad . \quad (4)$$

From (3) we see that a twisting moment of magnitude T_ε, unaccompanied by bending, would produce a torsional shear stress of magnitude f_n, and hence a principal stress of the same value, since the principal stress due to a shear has the same magnitude as the shear stress.

T_ε is sometimes called the "Equivalent Twisting Moment." It should be borne in mind, however, that it is not a twisting moment.

From (4) we see that a bending moment of magnitude M_ε, unaccompanied by torsion, would produce a direct bending stress equal in magnitude to the principal stress f_n. M_ε is called the " Equivalent Bending Moment."

EXAMPLE 4.

A shaft 3 in. diameter is subjected to a bending moment of 13 in. tons, and a torsion of 28 in. tons. Determine : (1) the maximum normal stress on a section perpendicular to the axis, (2) the maximum shear stress on a section perpendicular to the axis, (3) the two principal stresses, and (4) the maximum shear stress. (Lond. Univ., 1923.)

$$(1)\quad f = \frac{4M}{\pi r^3} = \frac{4 \times 13 \times 8}{\pi \times 27} = 4{\cdot}904 \text{ tons/sq. in.}$$

$$(2)\quad f_s = \frac{2T}{\pi r^3} = \frac{2 \times 28 \times 8}{\pi \times 27} = 5{\cdot}281 \text{ tons/sq. in.}$$

$$(3)\quad f_n = \tfrac{1}{2}\{f \pm \sqrt{f^2 + 4f_s^2}\}$$

$$= \tfrac{1}{2}\{4{\cdot}9 \pm \sqrt{(4{\cdot}9)^2 + 4(5{\cdot}28)^2}\}$$

$$= \tfrac{1}{2}\{4{\cdot}9 \pm \sqrt{24 + 112}\}$$

$$= \tfrac{1}{2}\{4{\cdot}9 \pm 11{\cdot}67\}$$

$= 8{\cdot}285$ tons/sq. in. (tension) and
$3{\cdot}38$ tons/sq. in. (compression)

(4) Maximum shear stress

$$= \frac{\sqrt{f^2 + 4f_s^2}}{2} = \frac{11{\cdot}67}{2}$$

$= 5{\cdot}835$ tons/sq. in.

115. **Application of Compound Stress Theories.** (*a*) *Maximum stress theory.*

If failure occurs due to the maximum principal stress, then we have that the stress causing failure is given by

$$f_n = \frac{2}{\pi r^3}\{M + \sqrt{M^2 + T^2}\}$$

or failure is produced by a torque of magnitude T_ε where

$$T_\varepsilon = M + \sqrt{M^2 + T^2} \quad . \quad . \quad . \quad . \quad (1)$$

(*b*) *Maximum strain theory.*

The stress corresponding to the maximum strain, and hence the stress causing failure, by this theory, is given by par. 24

$$= \frac{1}{2}\left\{f\left(1 - \frac{1}{m}\right) + \left(1 + \frac{1}{m}\right)\sqrt{f^2 + 4f_s^2}\right\}$$

$$= \frac{3}{8}f + \frac{5}{8}\sqrt{f^2 + 4f_s^2} \text{ when } m = 4$$

$$= \frac{3}{8}\left(\frac{4M}{\pi r^3}\right) + \frac{5}{8}\sqrt{\frac{16M^2}{(\pi r^3)^2} + \frac{16T^2}{(\pi r^3)^2}}$$

$$= \frac{2}{\pi r^3}\left\{\frac{3}{4}M + \frac{5}{4}\sqrt{M^2 + T^2}\right\}$$

or failure is produced by a torque of magnitude

$$T_\varepsilon = \frac{3}{4}M + \frac{5}{4}\sqrt{M^2 + T^2} \quad \quad (2)$$

(*c*) *Maximum shear stress theory.*

The shear stress causing failure will be

$$= \frac{\sqrt{f^2 + 4f_s^2}}{2}$$

$$= \frac{1}{2}\sqrt{\frac{16M^2}{(\pi r^3)^2} + \frac{16T^2}{(\pi r^3)^2}}$$

$$= \frac{2}{\pi r^3}\sqrt{M^2 + T^2}$$

Hence, a torque of magnitude T_ε will cause failure where

$$T_\varepsilon = \sqrt{M^2 + T^2} \quad \quad (3)$$

(*d*) *Maximum strain energy theory.*

The greatest strain energy is given by (1), par. 28, as

$$W = \frac{1}{2E}\left\{f_{n1}^2 + f_{n2}^2 - \frac{2f_{n1}f_{n2}}{m}\right\}$$

where
$$f_{n1} = \tfrac{1}{2}\{f + \sqrt{f^2 + 4f_s^2}\},$$
$$f_{n2} = \tfrac{1}{2}\{f - \sqrt{f^2 + 4f_s^2}\}$$

If f_v is the yield stress, then

$$\frac{f_v^2}{2E} = \frac{1}{2E}\left\{f_{n1}^2 + f_{n2}^2 - \frac{2f_{n1}f_{n2}}{m}\right\}$$

$$\therefore \quad f_v^2 = f^2 + \frac{2(m+1)}{m} f_s^2$$

$$= \frac{16M^2}{(\pi r^3)^2} + \frac{2(m+1)}{m}\frac{4T^2}{(\pi r^3)^2}$$

The torque T_ε, acting alone, which would bring the material to the point of failure, is found from

$$f_v^2 = 0 + \frac{2(m+1)}{m}\frac{4T_\varepsilon^2}{(\pi r^3)^2}$$

$$\therefore \quad \frac{2(m+1)}{m}\frac{4T_\varepsilon^2}{(\pi r^3)^2} = \frac{16M^2}{(\pi r^3)^2} + \frac{2(m+1)}{m}\frac{4T^2}{(\pi r^3)^2}$$

$$\therefore \quad T_\varepsilon = \sqrt{\frac{2m}{m+1}M^2 + T^2} \quad . \quad . \quad . \quad (4)$$

(1) Is usually referred to as the "Rankine" formula, and (3) as the "Guest" formula. With the present state of our knowledge of complex stresses, the "Guest" formula will be the most correct for shafts of ductile material.

116. **Torsion combined with Bending, Shear and Thrust.**

Let M = the bending moment
T = ,, torque
W = ,, shearing force due to bending
P = ,, end thrust

Then for a solid shaft of radius r, we have

Maximum compression stress $= \dfrac{P}{\pi r^2} + \dfrac{4M}{\pi r^3}$ . . (1)

Minimum ,, ,, $= \dfrac{P}{\pi r^2} - \dfrac{4M}{\pi r^3}$ . . . (2)

The shear stress due to W is zero at top and bottom of the shaft, and maximum at a horizontal diameter $= \dfrac{4}{3}\dfrac{W}{\pi r^2}$.

Total shear stress at top or bottom

$$=\frac{2T}{\pi r^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Total shear stress at each end of a horizontal diameter

$$=\frac{2T}{\pi r^3}+\frac{4W}{3\pi r^2} \quad . \quad . \quad . \quad . \quad (4)$$

The principal stresses are found in the usual way from

$$f_n = \tfrac{1}{2}\{f \pm \sqrt{f^2 + 4f_s^2}\}$$

For the top or bottom of the shaft

$$f = \frac{P}{\pi r^2} \pm \frac{4M}{\pi r^3} \text{ and } f_s = \frac{2T}{\pi r^3}$$

For each end of a horizontal diameter the bending stress is zero, and

$$f = \frac{P}{\pi r^2} \text{ and } f_s = \frac{2T}{\pi r^3} + \frac{4W}{3\pi r^2}$$

For torsion and thrust only

$$f = \frac{P}{\pi r^2} \text{ and } f_s = \frac{2T}{\pi r^3}$$

For torsion thrust and shear

$$f = \frac{P}{\pi r^2} \text{ and } f_s = \frac{2T}{\pi r^3}$$

for the top or bottom of the shaft, and at each end of a horizontal diameter

$$f_s = \frac{2T}{\pi r^3} + \frac{4W}{3\pi r^2}$$

For bending, torsion and shear

$$f = \frac{4M}{\pi r^3}$$

at top and bottom of shaft, and

$$f_s = \frac{2T}{\pi r^3}$$

and at each end of a horizontal diameter

$$f = 0 \text{ and } f_s = \frac{2T}{\pi r^3} + \frac{4W}{3\pi r^2}$$

117. Close-coiled Helical Springs. A helical spring, such as that in Fig. 89, carrying an axial load, has every cross-section subjected to torsion, and to a bending moment tending to alter the curvature of the coils; there is also a shear stress due to the axial load. When the coils are closely wound, the bending stress is negligible in comparison to the torsional stress, and in nearly all cases the shear stress is unimportant.

Suppose a helical spring to have n coils, then the total length of wire is given, approximately, by

$$l = 2\pi R n$$

where R is the mean radius of the coils.

The twisting moment due to the *axial load* is $WR = T$. Then if θ is the angle of twist of the wire, and δ the axial deflection, we obtain, by equating the work done by W to the resilience of the spring, that

$$\tfrac{1}{2}W\delta = \tfrac{1}{2}T\theta$$

or $$\delta = \frac{T}{W}\theta$$

FIG. 89

but $$\theta = \frac{Tl}{I_p C} = \frac{WR \times 2\pi R n}{\frac{\pi r^4}{2} \times C} = \frac{4WR^2 n}{Cr^4}$$

$$\therefore \quad \delta = \frac{WR}{W} \times \frac{4WR^2 n}{Cr^4}$$

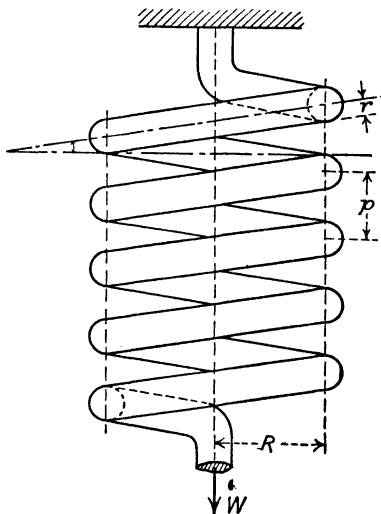
$$= \frac{4WR^3 n}{Cr^4} \qquad \qquad (1)$$

or $$\delta = \frac{64WR^3 n}{Cd^4} \qquad \qquad (2)$$

where d is the diameter of the wire.

The resilience of the spring

$$= \tfrac{1}{2}T\theta$$

$$= \frac{1}{2} WR \times \left(\frac{f_s}{r}\frac{l}{C}\right)$$

$$= \frac{1}{2}\left(\frac{\pi r^4}{2}\frac{f_s}{r}\right)\frac{f_s}{r}\frac{l}{C}$$

$$= \frac{1}{4}\frac{f_s^2}{C}(\pi r^2 l)$$

$$= \frac{1}{4}\frac{f_s^2}{C}(\text{volume of wire}) \quad . \quad . \quad . \quad (3)$$

The "stiffness" of the spring, which is the force per unit deflection, $= \frac{1}{\delta}$ when the axial force is 1 lb.

In the case of a spring constructed of wire, of square section of side x, we have, by par. 113, (3)

$$\delta = \frac{T}{W} \times \frac{Tl}{C}\frac{42I_p}{A^4}$$

$$= R \times \frac{WRl}{C} \times \frac{42x^4}{6x^8}$$

$$= 7\frac{WR^2l}{Cx^4} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The resilience will be the same as that obtained in par. 113 (5).

If a close-coiled spring be subjected to an *axial torque*, then there is a constant bending moment acting on the coils of magnitude equal to that of the torque. This bending moment will increase or decrease the curvature of the coils, depending on its direction of action.

Assuming each coil to act as a beam of large curvature, we have by par. 74 (7), that the resilience

$$= \frac{M^2L}{2IE}$$

then, if ϕ is the angle of twist of the free end,

$$\tfrac{1}{2}M\phi = \frac{M^2L}{2IE}$$

$$\phi = \frac{ML}{IE} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$= \frac{8MRn}{Er4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or $$\phi = \frac{128MRn}{Ed^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{The resilience} = \frac{f^2I^2}{r^2}\,\frac{L}{2IE} = \frac{f^2}{2E}\,\frac{\pi r^2L}{4}$$

$$= \frac{1}{8}\frac{f^2}{E}\text{ (volume of wire)} \quad . \quad . \quad . \quad (8)$$

For wire of square section of side x, we have

$$\phi = \frac{24\pi RMn}{Ex^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and resilience $$= \frac{1}{6}\frac{f^2}{E}\text{ (volume of wire)} \quad . \quad . \quad . \quad (10)$$

EXAMPLE 5.

A closely-coiled helical spring is to be made having coils each 4 in. mean diameter from wire of $\frac{1}{4}$ in. diameter and of such a stiffness that it will elongate axially 1 in. for an axial pull of 20 lb. How many coils should the spring have if the modulus of rigidity of the material is 11,500,000 lb./sq. in. ? (A.M.I.Mech.E., 1926.)

$$\delta = \frac{64WR^3n}{Cd^4}$$

$$\therefore \qquad n = \frac{Cd^4\delta}{64WR^3}$$

$$= \frac{11{\cdot}5 \times 10^6 \times 1}{16 \times 16 \times 64 \times 20 \times 8}$$

$$= 4{\cdot}4 \text{ coils}$$

118. **Open-coiled Helical Springs.** If p is the pitch of the coils, and a the constant angle which the coils make with planes

perpendicular to the axis, then $\tan a = \frac{p}{2\pi R}$ and the length of wire $= \frac{2\pi Rn}{\cos a}$.

Taking a section through the coils perpendicular to plane of the paper, and passing through the axis of the helix, and considering a portion of a coil, as in Fig. 90, where AB is the helix formed by the axis of the wire, the torque T, due to the *axial load*, is represented by the line AD, where AD is the axis about which the torque acts, at the section passing through A. It should be noted that AD is a tangent, at A, to the cylinder fitting inside the helix formed by the axis of the wire.

FIG. 90

Take an axis XX, tangential to AB at A, and also tangential to the cylinder, and take YY, perpendicular to XX, at A, and also tangential to the cylinder, then AD, XX, and YY lie in the same plane.

The torque $T = WR$ can be resolved into a component $T \cos a$ along $XX = WR \cos a$, which is at all sections causing torsion about the axis of the wire, and a component $T \sin a$ about $YY = WR \sin a$, which is causing torsion about the axis of the helix, i.e. a bending moment on the coils.

Equating the external work to the resilience in torsion and bending, we have

$$\tfrac{1}{2}W\delta = \tfrac{1}{2}(WR \cos a)\theta + \tfrac{1}{2}(WR \sin a)\phi$$

or

$$\delta = R[(\cos a)\,\theta + (\sin a)\phi]$$

$$= R\left[(\cos a)\,\frac{T \cos al}{I_p C} + (\sin a)\,\frac{Ml}{IE}\right]$$

$$= R\left[\cos^2 a \frac{WRl}{I_pC} + \frac{WR\sin^2 al}{IE}\right]$$

$$= WR^2l\left[\frac{\cos^2 a}{I_pC} + \frac{\sin^2 a}{IE}\right]$$

$$= \frac{2\pi WR^3n}{\cos a}\left[\frac{\cos^2 a}{I_pC} + \frac{\sin^2 a}{IE}\right]$$

$$= \frac{4WR^3n}{r^4\cos a}\left[\frac{\cos^2 a}{C} + \frac{2\sin^2 a}{E}\right] \quad . \quad . \quad (1)$$

or

$$\delta = \frac{64WR^3n}{d^4\cos a}\left[\frac{\cos^2 a}{C} + \frac{2\sin^2 a}{E}\right] \quad . \quad . \quad (2)$$

The deflection of a spring of square section may be found by using the value of θ in par. 113 (3). This gives

$$\delta = \frac{WR^2l}{x^4}\left[\frac{7\cos^2 a}{C} + \frac{12\sin^2 a}{E}\right]$$

$$= \frac{2\pi WR^3n}{x^4\cos a}\left[\frac{7\cos^2 a}{C} + \frac{12\sin^2 a}{E}\right] \quad . \quad (3)$$

When the open coiled spring is subjected to an *axial torque*, the torque may be resolved into components $M_1\cos a$ about YY, and $M_1\sin a$ about XX (see Fig. 90), the former giving a constant bending moment tending to alter the curvature of the coils, and the latter a twisting moment about the axis of the wire. If ϕ is the angle which the free end moves through

$$\frac{1}{2}M_1\phi = \frac{1}{2}M_1\cos a\left(\frac{M_1\cos al}{EI}\right) + \frac{1}{2}M_1\sin a\left(\frac{M_1\sin al}{CI_p}\right)$$

$$\phi = M_1l\left[\frac{\cos^2 a}{EI} + \frac{\sin^2 a}{CI_p}\right]$$

$$= \frac{2\pi RM_1n}{\cos a}\left[\frac{\cos^2 a}{EI} + \frac{\sin^2 a}{CI_p}\right]$$

$$= \frac{4RM_1n}{r^4\cos a}\left[\frac{2\cos^2 a}{E} + \frac{\sin^2 a}{C}\right] \quad . \quad . \quad . \quad (4)$$

$$\text{or } \phi = \frac{64RM_1n}{d^4\cos a}\left[\frac{2\cos^2 a}{E} + \frac{\sin^2 a}{C}\right] \quad . \quad . \quad . \quad (5)$$

For a spring of square section of side x, by substituting the values of I and I_p, we have

$$\phi = \frac{2\pi M_1 Rn}{x^4 \cos a}\left[\frac{12\cos^2 a}{E} + \frac{7\sin^2 a}{C}\right] \quad . \quad . \quad . \quad (6)$$

EXAMPLE 6.

An open-coil spiral spring consists of ten coils, each of a mean diameter of 2 in., the wire forming the coils being $\frac{1}{4}$ in. diameter, and making a constant angle of 30° with planes perpendicular to the axis of the spring. What load would cause the spring to elongate $\frac{1}{2}$ in., and what are the bending and shear stresses due to this load ? Calculate the value of the axial twist which would cause a bending stress of 8,000 lb./sq. in. in the coils. $E = 30 \times 10^6$ lb./sq. in., $C = 12 \times 10^6$ lb./sq. in.

$$\delta = \frac{64WR^3n}{d^4 \cos a}\left[\frac{\cos^2 a}{C} + \frac{2\sin^2 a}{E}\right]$$

$$0{\cdot}5 = W\frac{64 \times 1 \times 10 \times 16 \times 16}{0{\cdot}866}\left[\frac{(0{\cdot}866)^2}{12 \times 10^6} + \frac{2(0{\cdot}5)^2}{30 \times 10^6}\right]$$

$$0{\cdot}5 = W\frac{64 \times 10 \times 16 \times 16}{0{\cdot}866} \times \frac{1}{3 \times 10^6} \times \frac{4{\cdot}75}{20}$$

$$\therefore W = \frac{0{\cdot}5 \times 0{\cdot}866 \times 3 \times 10^6 \times 20}{64 \times 10 \times 16 \times 16 \times 4{\cdot}75}$$

$$= 33{\cdot}4 \text{ lb.}$$

$$\text{Bending moment} = WR\sin a = 33{\cdot}4 \times 1 \times 0{\cdot}5$$

$$= 16{\cdot}7 \text{ lb. in.}$$

$$f = \frac{4M}{\pi r^3} = \frac{32M}{\pi d^3} = \frac{32 \times 16{\cdot}7 \times 64}{\pi}$$

$$= 10{,}900 \text{ lb./sq. in.}$$

Twisting moment about axis of wire

$$= WR\cos a = 33{\cdot}4 \times 0{\cdot}866 \text{ lb. in.}$$

$$f_s = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3} = \frac{16 \times 33{\cdot}4 \times 0{\cdot}866 \times 64}{\pi}$$

$$= 9420 \text{ lb./sq. in.}$$

Let M_1 be the axial torque.

Component of axial torque causing bending

$$= M_1 \cos a$$

$$\therefore \qquad 8000 = \frac{M_1 \cos a \times 32}{\pi d^3}$$

$$\therefore \qquad M_1 = \frac{\pi \times 8000}{64 \times 32 \times 0{\cdot}866}$$

$$= 14{\cdot}25 \text{ lb. in.}$$

EXAMPLES XI

1. Determine the diameter of a solid steel shaft which will transmit 450 h.p. at 300 r.p.m. The twist must not exceed 1 degree in 7 ft. length, nor the maximum shear stress 5,500 lb./sq. in. Modulus of rigidity 11,500,000 lb./sq. in. (Lond. Univ., 1922.) *Ans.*, 4·44 in. or $4\frac{1}{2}$ in. approx.

2. A steel shaft is required to transmit 80 h.p. at 60 r.p.m., and the maximum twisting moment is 30 per cent greater than the mean. Find the diameter for a shear stress of 8,000 lb. per square inch, also the twist of the shaft on a length of 10 ft. $C = 5{,}200$ tons per square inch. (A.M.I.Mech.E., 1919.) *Ans.*, $d = 4{\cdot}1$ in. $\theta = 2{\cdot}35°$.

3. A vessel having a single propeller shaft 12 in. in diameter, and running at 160 r.p.m., is re-engined with turbines driving two equal propeller shafts at 750 r.p.m. and developing 60 per cent more horse-power. If the working stresses of the new shafts are 10 per cent greater than that of the old shaft, find their diameters. (Lond. Univ., 1913.) *Ans.*, 6·45 in. or $6\frac{1}{2}$ in. approx.

4. In calculating the size of shafts to transmit power, the following formula is sometimes used,

$$d = \sqrt[3]{\frac{60 \times \text{horse-power}}{\text{revs. per min.}}}$$

What shear stress is allowed in the shaft by this rule ? A shaft is to be made up of two lengths connected by a flanged coupling taking 8 bolts on a pitch circle diameter which may be assumed to be $2\frac{1}{2}$ in. greater than the shaft diameter. If 400 h.p. is to be transmitted at 80 r.p.m., determine suitable diameters of shafts and bolts. The shear stress in the bolts is to be 20 per cent in excess of that in the shaft, which is to be fixed by the above rule. (Lond. Univ., 1921.) *Ans.*, $6\frac{3}{4}$ in. and 1 in. approx.

5. A shaft is 5 ft. long, 3 in. diameter at one end, and tapers at a uniform rate to 4 in. diameter at the other end. The larger end is firmly fixed and a torque of 2,500 lb. ft. is applied to the smaller end. Find the total angle of twist and the maximum shear stress. Assume a modulus of torsion of 11·5 millions. (Lond. Univ., 1921.) *Ans.*, $\theta = 0{\cdot}652°$. $f = 5{,}650$ lb./sq. in.

6. Find the size of a square shaft which will transmit 100 h.p. at 120 r.p.m. The shear stress is not to exceed 3 tons per square inch. (London Univ., 1918.) *Ans.*, $3\frac{3}{8}$ in. × $3\frac{3}{8}$ in.

7. Calculate the dimensions of a hollow steel shaft to transmit 2,100 h.p. at a speed of 120 r.p.m.; the maximum twisting moment being 1·25 times the mean. The internal diameter of the shaft is 60 per cent of the outside diameter, and the greatest intensity of shear stress in the steel is limited to 3 tons/sq. in. (A.M.I.Mech.E., 1924.) *Ans.*, $D = 10\frac{5}{8}$ in., $d = 6\frac{3}{8}$ in.

8. In determining the horse-power transmitted by a turbine-driven propeller shaft by means of a torsion meter, it is found that the angle of twist, measured on a length of 20 ft., is 1·2 degrees. The external and internal diameters of the shaft are 10 in. and 7 in. respectively, and the speed is 250 r.p.m. Find the horse-power being transmitted by the shaft, and the maximum intensity of shear stress induced in the material. $C = 5{,}200$ tons/sq. in. (A.M.I.Mech.E., 1924.) *Ans.*, 3,039 h.p., 5,150 lb./sq. in.

9. Define "principal stresses." A solid shaft, 16 in. diameter, is subjected to a bending moment of 120 ft. tons, and a twisting moment of 50 ft. tons. Find the maximum tensile, compressive and shear stresses in the shaft. Find also the maximum tensile strain and the corresponding stress. Poisson's ratio = 0·3, $E = 13{,}000$ tons/sq. in. (Lond. Univ., 1918.)
Ans., 8,350 and 4,337 lb./sq. in. respectively, 0·00029, 8,460 lb./sq. in.

10. A shaft is required for an engine which indicates 1,000 h.p. at 80 r.p.m The maximum twisting moment on the shaft is 1·8 times the mean twisting moment. The main bearings are 15 ft. apart, and the shaft carries a flywheel midway between bearings weighing 90 tons, the bending moment due to this weight is additional to that due to the steam pressure, the latter of which is 0·8 times the mean twisting moment. Find the diameter of the shaft to satisfy the condition that the maximum tensile stress in the material is 4 tons/sq. in. (Lond. Univ., 1913.) *Ans.*, $22\frac{1}{4}$ in.

11. A cylindrical steel shaft, 1 in. in diameter, is subjected to a pull of 8,000 lb. along its axis, and to a twisting moment of 200 lb. ft. about its axis. Find the greatest intensity of stress in the material. (A.M.I.Mech.E., 1926.) *Ans.*, 18,300 lb./sq. in.

12. A propeller of 6 tons weight is carried by a shaft of 9 in. diameter, and overhangs the supporting bracket by 18 in. The propeller received 4,000 h.p. at a speed of 300 r.p.m. If the propeller thrust is 15 tons, calculate the principal stresses at a point on the surface of the shaft, and draw a diagram showing how these principal stresses at that point vary throughout a complete revolution. (Lond. Univ., 1921.)
Ans., At bottom 8,134 lb./sq. in., at end of horizontal dia., 6,424 lb./sq. in.

13. A close-coiled helical spring is made from steel wire 0·25 in. diameter, and there are ten free coils having a mean diameter of 3 in. The spring is subjected to an axial load of 20 lb. Determine the maximum intensity of shear stress in the steel, and the total deflection under the above load. What is the stiffness of the spring in pounds per foot of deflection? $C = 12 \times 10^6$ lb./sq. in. (A.M.I.Mech.E., 1924.)
Ans., 9,780 lb./sq. in., 0·9215 in., 260 lb.

14. A spiral spring which is to be used in compression is made of steel rod $\frac{1}{4}$ in. diameter. The mean diameter of the spring is 5 in., there are 18 coils, and when unloaded the spring is 10 in. long. It is placed with its axis vertical. Find the gradually applied load which will compress the spring 1 in. If this load is allowed to drop from a point 3 in. above the top of the unloaded spring, find the maximum compression produced in the spring and the maximum shear stress in the material. $C = 12 \times 10^6$ lb./sq. in. Neglect the mass of the spring. (Lond. Univ., 1925.)
Ans., 2·6 lb., 3·64 in., 7,730 lb./sq. in.

15. A closely coiled helical spring, made of wire 0·2 in. in diameter and having an inside diameter of $1\frac{1}{2}$ in., joins two shafts. The effective number of coils between the shafts is 15, and 1 h.p. is transmitted through the spring at 1,000 r.p.m. Calculate the relative axial twist in degrees between the ends of the spring, and also the intensity of bending stress in the material. $E = 30 \times 10^6$ lb./sq. in. (A.M.I.C.E., 1925.)

Ans., 2·15 radians, 80,500 lb./sq. in.

16. A closed coil cylindrical spiral spring of circular section has coils 3 in. mean diameter. When loaded with an axial load of 55 lb. it is found to extend 5·4 in., and when subjected to a twisting couple about the axis of the spring of 30 lb. in. there is an angular rotation of 60 degrees. Determine Poisson's Ratio for the material. (Lond. Univ., 1935.)

Ans., $\frac{1}{4}$.

17. A straight rod, length l, diameter d, is subjected to a twisting moment, T, about the axis of the rod. A similar rod is wound into a close coiled spring and subjected to a twisting moment, T, about the axis of the spring. Compare the strain energies.

An extension spindle, $\frac{1}{2}$ in. diameter, 30 in. long, drives a recording instrument through a close coiled spring of 7 coils, $1\frac{1}{2}$ in. mean diameter. Determine (*a*) the wire diameter, if the strain energy of the spring is to be 100 times that of the spindle, and (*b*) the total angle of twist produced by a suddenly applied torque of 5 lb. in. $E = 30 \times 10^6$ lb./sq. in. $C = 12 \times 10^6$ lb./sq. in. (Lond. Univ., 1933.)

Ans., $\frac{1·25}{1}$, 0·153 in., 23·6°.

18. In an open-coil helical spring of 10 coils the stresses due to bending and twisting are 14,000 and 15,000 lb./sq. in. respectively when the spring is axially loaded. Assuming the mean diameter of the coils to be eight times the diameter of the wire, find the maximum permissible axial load and the diameter of the wire for a maximum extension of 0·7 in. $E = 30 \times 10^6$ lb./sq.in. $C = 11 \times 10^6$ lb./sq. in. (Lond. Univ., 1940.) *Ans.*, 39 lb., 0·22 in.

CHAPTER XII

COLUMNS AND STRUTS

119. All pieces of material subjected to compression are included under the above heading, and these may be arranged into three classes, viz.—

1. Those having a very short length.
2. Those not greater in length than about thirty diameters.
3. Those of greater length than about thirty diameters.

Each class can be further sub-divided into (a) those having axial loading, and (b) those in which the loading is eccentric.

Columns under class (1) have already been dealt with, section (a) being found in par. 5, and section (b) in par. 72. This chapter will be devoted to the investigation of columns in classes (2) and (3); in each case axial and eccentric loading being considered, and it will be assumed that the column is of uniform cross-section.

120. **Euler's Theory of Long Columns.** *Rounded Ends.* The column shown in Fig. 91B carries an axial load P. Its ends are said to be "rounded," or "pin-jointed," in that they are perfectly free to change their slope, but all other movement is prevented.

Suppose P to be the "buckling load" of the column, that is the load is such that if the column receives slight lateral displacement, as shown, then the elastic forces tending to straighten the column will just be able to balance the effect of P. Let O be the mid-point of the length, then the bending moment at a point on the column distant y from O is given by

$$M = Px$$

but $$-\frac{d^2x}{dy^2} = \frac{M}{EI} \text{ where } I \text{ is the } \textit{least} \text{ moment of inertia}$$

$$\therefore \quad \frac{d^2x}{dy^2} = -\frac{P}{EI}x$$

$$= -k^2x \text{ where } k^2 = \frac{P}{EI}$$

or $$\frac{d^2x}{dy^2} + k^2x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The solution of (1) is given by

$$x = A \cos ky + B \sin ky \quad . \quad . \quad . \quad (2)$$

where A and B are constants.

Differentiating (2), we have

$$\frac{dx}{dy} = -Ak \sin ky + Bk \cos ky$$

and at O, where $y = 0$, $\frac{dx}{dy} = 0$, and $\sin \{k(0)\} = 0$

$$\therefore 0 = Bk \cos \{k(0)\}$$

or

$$B = 0, \text{ and } x = A \cos ky \quad . \quad . \quad . \quad (3)$$

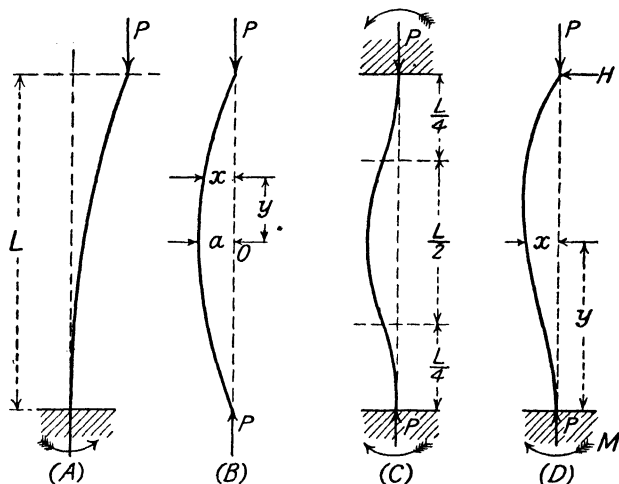



Fig. 91

Also, when $y = 0$, $x = a$, and $\cos (k(0)) = 1$, $\therefore A = a$, and when $x = 0$, $y = \frac{L}{2}$.

Hence $0 = A \cos k \frac{L}{2}$, and, since it has just been shown that A is not zero, then $\cos k \frac{L}{2} = 0$.

The solution of which is given by

$$k \frac{L}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc.}$$

The smallest value of P is found from

$$k\frac{L}{2} = \frac{\pi}{2} \text{ or } k^2\frac{L^2}{4} = \frac{\pi^2}{4}$$

$$\therefore \qquad \frac{P}{EI}L^2 = \pi^2$$

and $$P = \frac{\pi^2 EI}{L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The buckling loads for cases A and C can be found in a similar manner, but a solution is obtained more rapidly by considering the " equivalent length " of the column, having rounded ends.

One End "Fixed," the other End "Free." This is case A, and the equivalent length of the column having rounded ends is $2L$. Substituting this in (4), we have the buckling load given by

$$P = \frac{\pi^2 EI}{(2L)^2}$$

or $$P = \frac{\pi^2 EI}{4L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The strength of this column is thus only one-fourth of that for a similar column having rounded ends.

Both Ends Fixed. This is represented by case C, Fig. 91, and an examination of the figure shows that the equivalent length of a column having rounded ends is given by $\frac{L}{2}$.

Substituting this value in (4), the buckling load for this case is given by

$$P = \frac{\pi^2 EI}{\left(\frac{L}{2}\right)^2}$$

or $$P = \frac{4\pi^2 EI}{L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This column is four times as strong as that represented by case B, and sixteen times as strong as that represented by case A. Great care must be taken in deciding whether a column is " pin-jointed " or " fixed." In practice, very few cases of absolute fixing are met with, as the great majority come under

the heading of "pin-jointed" ends, or are mid-way between the two cases.

One End Fixed, other End Free to Rotate. This is shown by case D, Fig. 91. The bending moment M is introduced by the "fixing," which involves the use of the horizontal force H for equilibrium.

At a distance y from the fixed end, let x be the amount of deviation from the perpendicular.

The bending moment at this point is given by

$$M = H(L - y) - Px$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{1}{EI}\{H(L - y) - Px\}$$

$$\text{or} \quad \frac{d^2x}{dy^2} + \frac{P}{EI}x = \frac{H}{EI}(L - y) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The solution of (7) is given by

$$x = A\cos\left\{\sqrt{\frac{P}{EI}}\cdot y\right\} + B\sin\left\{\sqrt{\frac{P}{EI}}\cdot y\right\} + \frac{H}{P}(L - y) \quad (8)$$

At $y = 0$, $x = 0$.

$$\therefore \quad 0 = A\cos\left\{\sqrt{\frac{P}{EI}}(0)\right\} + \frac{H}{P}L$$

$$\text{or} \quad 0 = A + \frac{H}{P}L$$

$$\therefore \quad A = -\frac{H}{P}L \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Differentiating (8), we have

$$\frac{dx}{dy} = -A\sqrt{\frac{P}{EI}}\sin\left\{\sqrt{\frac{P}{EI}}\cdot y\right\} + B\sqrt{\frac{P}{EI}}\left\{\cos\sqrt{\frac{P}{EI}}\cdot y\right\} - \frac{H}{P}$$

and at $y = 0$, $\frac{dx}{dy} = 0$.

$$\therefore \quad 0 = B\sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$\text{or} \quad B = \frac{H}{P}\sqrt{\frac{EI}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (8) may now be written in the form

$$x = -\frac{H}{P} L \cos\left\{\sqrt{\frac{P}{EI}} \cdot y\right\} + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left\{\sqrt{\frac{P}{EI}} \cdot y\right\} + \frac{H}{P}(L - y) \quad . \quad (11)$$

when $y = L$, $x = 0$, and hence

$$0 = -\frac{H}{P} L \cos\left\{\sqrt{\frac{P}{EI}} \cdot L\right\} + \frac{H}{P}\sqrt{\frac{EI}{P}} \sin\left\{\sqrt{\frac{P}{EI}} \cdot L\right\}$$

or

$$\tan\left\{\sqrt{\frac{P}{EI}} \cdot L\right\} = \sqrt{\frac{P}{EI}} \cdot L \quad . \quad . \quad (12)$$

The smallest value of P, from (12), (other than $P = 0$) is given by

$$\sqrt{\frac{P}{EI}} \cdot L = 4{\cdot}49$$

or

$$\frac{P}{EI} L^2 = (4{\cdot}49)^2 = 20$$

$$\doteqdot 2\pi^2$$

$\therefore$

$$P = 2\frac{\pi^2 EI}{L^2} \quad . \quad . \quad . \quad . \quad (13)$$

From (4), (5), (6), and (13), it will be observed that, according to the Euler theory, the buckling load for a column carrying an axial load is given by

$$P = K\frac{\pi^2 EI}{L^2} \quad . \quad . \quad . \quad . \quad (14)$$

	Case A	Case B	Case C	Case D
Value of K . .	$\frac{1}{4}$	1	4	2

EXAMPLE 1.

Give the proof of Euler's formula for the critical load of a strut with rounded ends. Using this formula, obtain the critical load of a strut which is made of a circular bar 10 ft. long. This bar, when freely supported at its ends, is found to deflect $\frac{1}{8}$ in. with a load of 10 lb. attached at the centre. (Lond. Univ., 1921.)

For a beam supported at each end and carrying a central load

$$\delta = \frac{1}{48}\frac{WL^3}{EI}$$

$$\therefore \qquad EI = \frac{1}{48} \cdot \frac{WL^3}{\delta}$$

Also, from par. 119

$$P = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2}{L^2} \times \frac{WL^3}{\delta} \times \frac{1}{48}$$

$$= \frac{\pi^2 L}{\delta} \times \frac{W}{48}$$

$$= \frac{\pi^2 \times 120}{0{\cdot}5} \times \frac{10}{48}$$

$$= 493{\cdot}7 \text{ lb.}$$

121. The Euler formula for a column or strut is only applicable when the length is great in comparison to the cross-sectional dimensions, and, in the case of a circular column, a rough rule may be taken to be that the length must be greater than thirty diameters ; the limit for any particular case being reached when the buckling stress is equal to the yield stress. The Euler theory neglects the direct stress in the column, and, for long columns, this is negligible in comparison to the buckling stress. Many cases are met, however, where both stresses are of importance. Various empirical formulae have been produced to meet cases of this kind, the following being representative of these—

Rankine Gordon Formula.

$$P = \frac{f_c A}{1 + a\left(\frac{L}{k}\right)^2} \qquad \qquad (1)$$

where P = the crippling load (tons).

f_c = the intensity of stress at the yield point in compression (tons/sq. in.).

A = area of the cross-section (sq. in.).

L = length of column (inches).

k = least radius of gyration of cross-section (inch-units).

a = a constant depending on the end conditions and on the material.

The safe load is obtained by dividing P by a suitable factor of safety.

Material	f tons/sq. in.	VALUES OF a			
		A	B	C	D
Mild steel . .	21·00	$\frac{16}{30,000}$	$\frac{4}{30,000}$	$\frac{1}{30,000}$	$\frac{16}{9 \times 30,000}$
Wrought iron .	16·00	$\frac{16}{36,000}$	$\frac{4}{36,000}$	$\frac{1}{36,000}$	$\frac{16}{9 \times 36,000}$
Cast iron . .	36·00	$\frac{16}{6,400}$	$\frac{4}{6,400}$	$\frac{1}{6,400}$	$\frac{16}{9 \times 6,400}$
Timber . .	2·25	$\frac{16}{3,000}$	$\frac{4}{3,000}$	$\frac{1}{3,000}$	$\frac{16}{9 \times 3,000}$

The Rankine Gordon formula may be written in the form

$$P\left(1 + a\left(\frac{L}{k}\right)^2\right) = f_c A$$

or
$$f_c = \frac{P}{A} + \frac{aL^2P}{Ak^2}$$

$$= \frac{P}{A} + \frac{aL^2P}{I} \quad . \quad . \quad . \quad (2)$$

The formula thus holds for very short struts where buckling stress is negligible, and gives $P = f_c A$. For long struts, where the direct stress is negligible,

$$P = \frac{f_c}{a}\frac{I}{L^2}$$

which is of the same form as the Euler formula, where

$$\frac{f_c}{a} = K\pi^2 E$$

EXAMPLE 2.

Calculate the buckling load of a wooden strut 2 in. square in section, 100 in. long, if one end is fixed and the other end free. E for wood = 2,000,000 lb. per square inch. What modification would you make in the calculation if

the strut were 30 in. long, and the known crushing stress of the wood was 4,000 lb. per square inch. (A.M.I.C.E., 1915.)

$$I = \frac{bd^3}{12} = \frac{2 \times 2^3}{12}$$

$$= \frac{4}{3} \text{ in. units}$$

$$P = \frac{\pi^2 EI}{4L^2}$$

$$= \frac{\pi^2 \times 2 \times 10^6 \times 4}{4 \times 10^4 \times 3}$$

$$= 658{\cdot}5 \text{ lb.}$$

When the length is 30 in., the Rankine Gordon formula would be used.

$$k^2 = \frac{I}{A} = \frac{4}{3 \times 4} = \frac{1}{3}$$

$$P = \frac{f_c A}{1 + a\left(\frac{L}{k}\right)^2}$$

$$= \frac{4000 \times 4}{1 + \frac{16}{3000}(900 \times 3)}$$

$$= \frac{16{,}000}{1 + 14{\cdot}4} = \frac{16{,}000}{15{\cdot}4}$$

$$= 1040 \text{ lb.}$$

EXAMPLE 3.

A steel stanchion is built of two rolled steel joists of I-section, 12 in. by 6 in. by ½ in., united by plates ⅜ in. thick and 16 in. wide, fastened to the flanges. The edges of the plates are flush with the outside edges of the joists. Using the Rankine formula for a strut, find the safe load for this stanchion if it is 24 ft. long, $f = 21$ tons per square inch, $a = \frac{1}{7500}$. (Lond. Univ., 1922.)

$$I_{xx} \text{ for joists} = 2\left\{\frac{6 \times 12^3}{12} - \frac{5{\cdot}5 \times 11^3}{12}\right\}$$

$$= 508 \text{ inch units}$$

$$I_{XX} \text{ for plates} = \frac{16 \times 13{\cdot}5^3}{12} - \frac{16 \times 12^3}{12} = \frac{4}{3}(13{\cdot}5^3 - 12^3)$$

$$= 976 \text{ inch units}$$

Total I_{XX} for section $= 508 + 976 = 1484$ inch units

$$I_{YY} \text{ for plates} = \frac{1{\cdot}5 \times 16^3}{12} = 512 \text{ inch units}$$

$$I_{YY} \text{ for joists} = 2\left\{\frac{1 \times 6^3}{12} + \frac{11 \times \frac{1^3}{2}}{12} + 11\tfrac{1}{2} \times 5^2\right\}$$

$$= 2\{306\}$$

$$= 612 \text{ inch units}$$

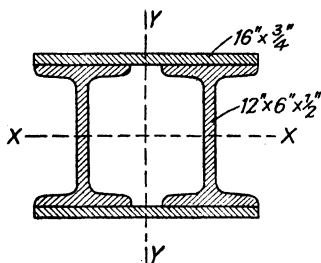



FIG. 92

Total I_{YY} for section $= 512 + 612 = 1124$ inch units.

Area of section $= 11{\cdot}5 + 11{\cdot}5 + 12 + 12 = 47$ sq. in.

$$\therefore \text{ Least value of } k^2 = \frac{1124}{47}$$

$$= 24$$

Crippling load for strut

$$= \frac{f_c A}{1 + a\left(\frac{L}{k}\right)^2}$$

$$= \frac{21 \times 47}{1 + \frac{1}{7500} \times \frac{(288)^2}{24}}$$

$$= \frac{21 \times 47}{1 \cdot 463}$$

$$= 676 \text{ tons}$$

With a factor of safety $= 4$

$$\text{Safe load} = \frac{676}{4}$$

$$= 169 \text{ tons}$$

122. **Straight Line Formula.** Formulae of this type are commonly used in America, and give results which are good enough for a rough approximation of the load. They are usually given in the form

$$P = fA\left(1 - x\frac{L}{k}\right)$$

where $P =$ the safe load in tons.

$f =$ the safe compressive stresses of a small length of the material in tons/sq. in.

A, L and k having the previous meaning.

$x =$ a constant depending on the material and on the method of constraint,

$x = 0 \cdot 005$ is a common value for mild steel,

and $x = 0 \cdot 008$,, ,, ,, cast iron.

123. **Parabolic Formula of Johnston.** This formula, which is intended to agree with the Euler formula for long columns, is given by

$$P = Af\left(1 - x\left(\frac{L}{k}\right)^2\right)$$

where x is a constant; the other letters have the same meanings as in par. 122.

With pin ends and $\frac{l}{k} < 150$

then for mild steel x may be taken as $0 \cdot 000023$.

124. **Fidler's Formula.** Fidler, in his *Bridge Construction*, has investigated the column problem, and he obtains the following formula—

$$P = \frac{A}{x}\left\{(f + H) - \sqrt{(f + H)^2 - 2xfH}\right\}$$

where P = the crippling or breaking stress in tons.
A = the cross-sectional area in sq. in.
f = the ultimate compressive strength of the material in tons/sq. in.
H = the Euler breaking stress $= \frac{\pi^2 E k^2}{L^2}$
x = a constant of average value 1·2.

125. "**Crinkling" of Columns.** A hollow mild steel column, whose thickness is small compared with its diameter, may "crinkle" or form into folds when under load. These folds may appear either just before or just after the direct stress is equal in magnitude to the yield stress of the material.

Southwell* in his investigation, obtains the formula

$$f = E\frac{t}{R}\sqrt{\frac{m^2}{3(m^2-1)}}$$

where f = the stress causing failure (tons/sq. in.).
t = the thickness of column (inches).
R = the mean radius of the column (inches).
E = Young's modulus for the material (tons/sq. in.).
$\frac{1}{m}$ = Poisson's ratio for the material.

Very few of the columns used in practice are liable to fail in this way, for, taking a mild steel whose yield stress is 21 tons/sq. in., E = 13,000 tons/sq. in., and $m = 4$, then $\frac{t}{R}$ must be less than $\frac{1}{370}$, in order that "crinkling" may occur.

125. **Columns with Eccentric Loads.** (A) Columns, other than those dealt with by the Euler formula, or those whose lengths are exceedingly small, when subjected to eccentric loading, may be treated as follows—

Let P = the safe load (Fig. 93).
y = the distance from the axis of the column to the line of action of the load.
A = the cross-sectional area of the column.
Z = the section modulus.

* *Phil. Trans. Roy. Soc.*, Vol. 213, Ser. A.

Then maximum stress due to bending

$$= \frac{P}{A} + \frac{M}{Z} \text{ where } M = Py$$

This stress must not exceed the safe stress obtained by one of the previous formulae. E.g. if f is the safe working stress according to the Rankine Gordon formulae, then

$$\frac{P}{A} + \frac{M}{Z} = f = \frac{f_c}{1 + a\left(\frac{L}{k}\right)^2}$$

or

$$\frac{P}{A}\left(1 + \frac{yx}{k^2}\right) = f$$

$$\therefore \qquad \frac{P}{A} = \frac{f}{1 + \frac{yx}{k^2}}$$

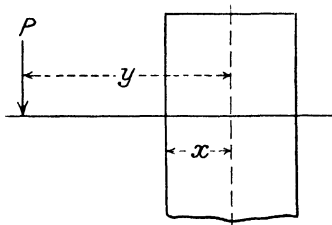



FIG. 93

or

$$\text{Safe eccentric load} = \frac{\text{safe central load}}{1 + \frac{yx}{k^2}}$$

It should be noted that the value of k^2, used in finding the safe central load, is the *least* value of k^2 for the section. The value of k^2 in the above equation is that used in calculating the moment of inertia required for the bending stress.

In the case of cast-iron columns subjected to eccentric loading, the cause of failure will probably be an excessive tensile stress. as cast iron is weak in tension. Thus, the safe load would be determined by the formula

$$\frac{P}{A} = \frac{f}{\frac{yx}{k^2} - 1}$$

where f is the safe tensile stress for cast iron.

Example 4.

Determine the safe load for the column given in Example 3, when the line of action of the load is at a point on YY distant 12 in. from the centre of the section.

From example 3, safe central load = 169 tons, and $k^2 = \frac{I}{A} = \frac{1494}{47}$

$$\text{Safe eccentric load} = \frac{\text{safe central load}}{1 + \frac{yx}{k^2}}$$

$$= \frac{169}{1 + \frac{12 \times 6{\cdot}75 \times 47}{1484}}$$

$$= \frac{169}{1 + 2{\cdot}57}$$

$$= \frac{169}{3{\cdot}57}$$

$$= 47{\cdot}34 \text{ tons.}$$

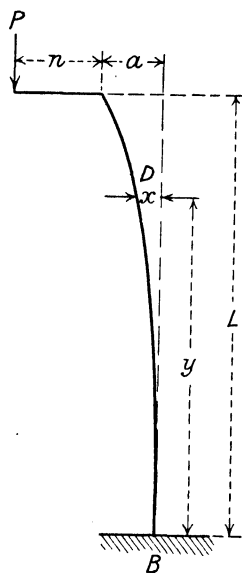



Fig. 94

(B) In the case of very long columns (Fig. 94) eccentrically loaded, the value of the safe load P may be found as follows—

Bending moment at $D = P(n + a - x)$

$$\therefore \quad EI\frac{d^2x}{dy^2} = P(n + a - x) \quad . \quad . \quad (1)$$

or

$$\frac{d^2x}{dy^2} = -\frac{P}{EI}(x - a - n)$$

hence $x - a - n = A \cos\sqrt{\frac{P}{EI}}\,y + B \sin\sqrt{\frac{P}{EI}}\,y \quad . \quad . \quad (2)$

at $y = 0$, $\frac{dx}{dy} = 0$, and $\therefore B$ must $= 0$.

$$\therefore \qquad x - a - n = A \cos\sqrt{\frac{P}{EI}}\,y$$

and at $y = 0$, $x = 0$

$\therefore A = -a - n$

and $$x - a - n = (-a - n)\cos\sqrt{\frac{P}{EI}}\,y$$

$$\therefore \qquad x = (a + n)\left(1 - \cos\sqrt{\frac{P}{EI}}\,y\right) \quad . \quad (3)$$

at $y = L$, $x = a$

and $$a = (a + n)\left(1 - \cos\sqrt{\frac{P}{EI}}\,L\right)$$

hence $$a = n\left(\sec\sqrt{\frac{P}{EI}}\,L - 1\right) \quad . \quad . \quad (4)$$

The bending moment at $B = M_B$

$$= P(n + a)$$

$$= Pn \sec\sqrt{\frac{P}{EI}}\,L \text{ from (4)}$$

and the bending stress at B

$$= \frac{M_B}{Z}$$

$$= \frac{Pn \sec\sqrt{\frac{P}{EI}}\,L}{Z}$$

$$= \frac{\left(Pn \sec\sqrt{\frac{P}{EI}}\,L\right)s}{Ak^2} \quad . \quad . \quad . \quad (5)$$

where s is half the depth of the section in the plane of bending, k^2 the radius of gyration of the section, and A the cross-sectional area.

$\therefore$ Maximum compressive stress at B

$$= \frac{P}{A} + \frac{Pns \sec\sqrt{\frac{P}{EI}}\,L}{Ak^2} \quad . \quad . \quad . \quad (6)$$

If f is the safe allowable compressive stress in the material then

$$f = \frac{P}{A}\left(1 + \frac{ns \sec \sqrt{\frac{P}{EI}}\, L}{k^2}\right) \quad . \quad . \quad (7)$$

$$\therefore \quad P = \frac{fA}{1 + \frac{ns \sec \sqrt{\frac{P}{EI}}\, L}{k^2}} \quad . \quad . \quad . \quad (8)$$

In the case of a cast-iron column, if f_t is the safe tensile stress for cast iron

$$\text{then} \quad f_t = \frac{P}{A}\left(\frac{ns \sec \sqrt{\frac{P}{EI}}\, L}{k^2} - 1\right)$$

$$\therefore \quad P = \frac{Af_t}{\frac{ns \sec \sqrt{\frac{P}{EI}}\, L}{k^2} - 1} \quad . \quad . \quad . \quad (9)$$

For a column having "free" ends, we may write $\frac{L}{2}$ for L in the previous equations, and thus

$$P = \frac{fA}{1 + \frac{ns \sec \sqrt{\frac{P}{EI}}\, \frac{L}{2}}{k^2}} \quad . \quad . \quad (10)$$

and for a cast-iron column

$$P = \frac{Af_t}{\frac{ns \sec \sqrt{\frac{P}{EI}}\, \frac{L}{2}}{k^2} - 1} \quad . \quad . \quad . \quad (11)$$

126. **Laterally-loaded Struts.** (*a*) A strut jointed at each end, at which the thrust is P, and carrying a lateral load W at mid-length is shown in Fig. 95.

The bending moment at $S = -Py - \frac{W}{2}\left(\frac{L}{2} - x\right)$

$$\therefore EI\frac{d^2y}{dx^2} = -Py - \frac{W}{2}\left(\frac{L}{2} - x\right) \quad . \quad . \quad . \quad . \quad (1)$$

the solution of which is of the form

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x + a + bx. \quad (2)$$

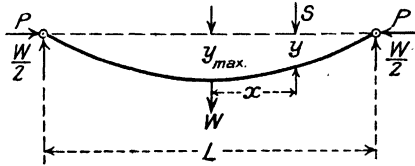



FIG. 95

Differentiating each side of (2)

$$\frac{d^2y}{dx^2} = -A\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}x - B\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}x \quad . \quad (3)$$

$$= -\frac{P}{EI}y - \frac{W}{2EI}\left(\frac{L}{2} - x\right) \text{ from (1)}$$

$$= -\frac{P}{EI}\left(A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x\right)$$

$$-\frac{P}{EI}(a + bx) - \frac{W}{2EI}\left(\frac{L}{2} - x\right) \quad . \quad . \quad . \quad (4)$$

$\therefore$ From (3) and (4)

$$-\frac{P}{EI}(a + bx) - \frac{W}{2EI}\left(\frac{L}{2} - x\right) = 0$$

from equating coefficients of x, $-\frac{P}{EI}b = -\frac{W}{2EI}$

$$\therefore \qquad b = \frac{W}{2P} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

equating constants, $-\frac{P}{EI}a = \frac{WL}{4EI}$

$$\therefore \qquad a = -\frac{WL}{4P} \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \quad y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x - \frac{WL}{4P} + \frac{W}{2P}x \quad (7)$$

at $x = 0, \frac{dy}{dx} = 0, \therefore B = -\frac{W}{2P}\sqrt{\frac{EI}{P}}$

hence
$$y = A\cos\sqrt{\frac{P}{EI}}\,x - \frac{W}{2P}\sqrt{\frac{EI}{P}}\sin\sqrt{\frac{P}{EI}}\,x - \frac{WL}{4P} + \frac{W}{2P}x \quad . \quad . \quad . \quad . \quad . \quad (8)$$

At $y = 0, x = \frac{L}{2}$

$$\therefore \quad 0 = A\cos\sqrt{\frac{P}{EI}}\frac{L}{2} - \frac{W}{2P}\sqrt{\frac{EI}{P}}\sin\sqrt{\frac{P}{EI}}\frac{L}{2} - \frac{WL}{4P} + \frac{WL}{4P}$$

and
$$A = \frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{L}{2}$$

$$\therefore \quad y = \frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{L}{2}\cos\sqrt{\frac{P}{EI}}\,x - \frac{W}{2P}\sqrt{\frac{EI}{P}}\sin\sqrt{\frac{P}{EI}}\,x - \frac{WL}{4P} + \frac{Wx}{2P} \quad . \quad (9)$$

At $x = 0$ y is maximum.

$$\therefore y_{max} = \frac{W}{2P}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{L}{2} - \frac{WL}{4P} \quad . \quad . \quad . \quad (10)$$

And maximum bending moment

$$= -Py_{max} - \frac{W}{2}\frac{L}{2}$$

$$= -\frac{W}{2}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{L}{2} \quad . \quad . \quad . \quad . \quad (11)$$

(*b*) In the case of a strut carrying a uniformly distributed lateral load of intensity w per unit length, we have

Bending moment at s (Fig. 96)

$$= -Py - \frac{wL}{2}\left(\frac{L}{2} - x\right) + \frac{w}{2}\left(\frac{L}{2} - x\right)^2$$

$$= -Py + \frac{wx^2}{2} - \frac{wL^2}{8} \quad . \quad . \quad . \quad . \quad (1)$$

and $$EI\frac{d^2y}{dx^2} = -Py + \frac{wx^2}{2} - \frac{wL^2}{8} \quad . \quad . \quad . \quad . \quad (2)$$

the solution of which is of the form

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$$

$$+ (a + bx + cx^2) \quad . \quad . \quad . \quad . \quad (3)$$

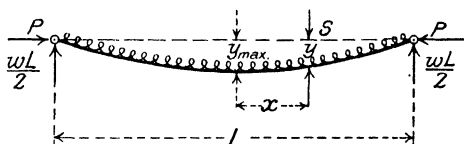



FIG. 96

Differentiating each side of (3)

$$\frac{d^2y}{dx^2} = -A\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}\,x - B\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}\,x + 2c \quad (4)$$

$$= -\frac{P}{EI}y + \left(\frac{wx^2}{2} - \frac{wL^2}{8}\right)\frac{1}{EI} \text{ from (2)}$$

$$= -\frac{P}{EI}\left\{A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x\right\}$$

$$-\frac{P}{EI}(a + bx + cx^2) + \frac{1}{EI}\left(\frac{wx^2}{2} - \frac{wL^2}{8}\right) \quad . \quad (5)$$

From (4) and (5)

$$2c = -\frac{P}{EI}(a + bx + cx^2) + \frac{1}{EI}\left(\frac{wx^2}{2} - \frac{wL^2}{8}\right)$$

Equating coefficients of x^2

$$-\frac{P}{EI}c + \frac{w}{2EI} = 0$$

$$\therefore \qquad c = \frac{w}{2P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equating coefficients of x, we get $b = 0$, and equating the constants

$$a = -\frac{wL^2}{8P} - \frac{wEI}{P^2} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\therefore \qquad y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x + \frac{w}{2P}x^2$$

$$-\frac{wL^2}{8P} - \frac{wEI}{P^2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

At $\qquad x = 0, \dfrac{dy}{dx} = 0$, and $\therefore B = 0$.

At $\qquad x = \dfrac{L}{2}, y = 0$, and hence

$$A = \frac{wEI}{P^2}\sec\sqrt{\frac{P}{EI}}\frac{L}{2}$$

The equation now becomes

$$y = \frac{wEI}{P^2}\sec\sqrt{\frac{P}{EI}}\frac{L}{2}\cos\sqrt{\frac{P}{EI}}\,x + \frac{w}{2P}x^2$$

$$-\frac{wL^2}{8P} - \frac{wEI}{P^2}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

At $x = 0$, y is maximum, and hence

$$y_{max} = \frac{wEI}{P^2}\sec\sqrt{\frac{P}{EI}}\frac{L}{2} - \frac{wL^2}{8P} - \frac{wEI}{P^2} \quad . \quad (10)$$

The maximum bending moment occurs where y is maximum, and is given by

$$M^{max} = -Py^{max} + \frac{wx^2}{2} - \frac{wL^2}{8}$$

$$= -P\left\{\frac{wEI}{P^2}\sec\sqrt{\frac{P}{EI}}\frac{L}{2} - \frac{wL^2}{8P} - \frac{wEI}{P^2}\right\} - \frac{wL^2}{8}$$

$$= \frac{wEI}{P}\left\{1 - \sec\sqrt{\frac{P}{EI}}\cdot\frac{L}{2}\right\} \quad . \quad . \quad . \quad (11)$$

EXAMPLE 5.

A horizontal strut consists of a steel bar of square section 1 in. × 1 in., and 12 ft. long. The bar carries an axial thrust of 5 cwt. Assuming $E = 30 \times 10^6$ lb./sq. in., estimate the maximum compressive stress in the bar.

$$I = \tfrac{1}{12} \text{ inch units}$$

$$w = 0{\cdot}28 \text{ lb. per inch length}$$

$$M_{max} = \frac{wEI}{P}\left(1 - \sec\sqrt{\frac{P}{EI}}\frac{L}{2}\right)$$

$$= \frac{0{\cdot}28 \times 30 \times 10^6 \times 1}{560 \times 12}\left(1 - \sec\sqrt{\frac{560 \times 12}{30 \times 10^6 \times 1}} \times 72\right)$$

$$= 1250\,(1 - \sec 1{\cdot}077)$$

$$= 1250\,(1 - 2{\cdot}06)$$

$$= -\,1325 \text{ lb. in.}$$

Maximum bending stress

$$= \frac{M}{I}y$$

$$= 1325 \times 12 \times 0{\cdot}5$$

$$= 7950 \text{ lb./sq. in.}$$

Stress due to direct thrust

$$= \frac{P}{A} = 560 \text{ lb./sq. in.}$$

∴ Maximum compressive stress in bar

$$= 7950 + 560$$

$$= 8510 \text{ lb./sq. in.}$$

EXAMPLES XII

1. Find the Euler crushing load for a hollow cylindrical cast-iron column 6 in. external diameter and $\frac{3}{4}$ in. thick, if it is 20 ft. long and hinged at both ends. $E = 5{,}000$ tons per square inch. Compare this load with the crushing load as given by the Rankine formula, using constants 36 and $\frac{1}{1600}$. For what length of column would these two formulae give the same crushing load? (Lond. Univ., 1926.) *Ans.*, 37·3 tons, 40·4 tons, 15·25 ft.

2. Obtain an expression for the load which will produce failure in a long strut axially loaded and both ends fixed. Discuss the truth of this expression in the case of hollow struts whose thicknesses are small compared with their diameters. (Lond. Univ., 1918.)

3. Compare the strength of columns 12 ft. long, containing the same volume of metal, (*a*) the column being rolled steel joist of I-section, 10 in. × 8 in. × $\frac{3}{4}$ in. ; (*b*) cast-iron hollow cylindrical column, the metal being $\frac{3}{4}$ in. thick.

Use Rankine's formula : $P = \dfrac{f_c A}{1 + a\left(\dfrac{l}{k}\right)^2}$

f_c for steel 21, for cast iron 36. a for steel $\frac{1}{7500}$, for cast iron $\frac{1}{1600}$.
(Lond. Univ., 1923.) *Ans.*, 215 tons, 243 tons.

4. A mild steel tube is 3 in. diameter and 0·1 in. thick. A short length of the tube is tested in compression and is found to yield at 35 tons per square inch. The modulus of elasticity of the material of the tube is 13,500 tons/sq. in. A length of 7 ft., when tested as a strut with free ends, failed with a load of 17 tons. Assuming the failing stress in the Rankine formula to be the yield stress of the material, find the denominator constant in the formula. Find also the crushing load as given by Euler's formula. (Lond. Univ., 1916.)
Ans., $\frac{1}{7724}$. 18 tons.

5. A stanchion is made of a 15 in. × 6 in. rolled steel joist with two plates 12 in. × $\frac{3}{8}$ in. riveted to each of the flanges. The web of the joist is 0·5 in. thick, and the flanges have a thickness of 0·88 in. The stanchion is 36 ft. long and is fixed at the ends. Find the safe load the stanchion will carry if it is axially loaded. Factor of safety 5. (Lond. Univ., 1915.)
Ans., 113·6 tons.

6. In the previous example, find the safe load if the line of action of the load lies on the centre line of the web and at a distance of 12 in. from the centre of gravity of the section. *Ans.*, 39 tons.

7. A hollow cast-iron column has an outer diameter of 12 in. and a thickness of 1 in., length 20 ft. Find the safe load which can be carried by the column when the line of action of the load is 6 in. from the centre of the column. Tensile strength of cast iron 9 tons/sq. in., factor of safety 6.
Ans., 38·3 tons.

8. Find the stress in the material of the hollow column in Q. 1, if the point of application of the load is 3 in. away from the centre of the section and the load carried is 9 tons. *Ans.*, 1·885 tons/sq. in.

9. The coupling-rod of a locomotive is 98 in. long, and has an I-section 5 in. deep. The area of the section is $4\frac{1}{2}$ sq. in., and its moment of inertia about a horizontal axis through the centre of gravity is 15 in. units. The lateral load on the rod, due to its inertia at full speed, is 276 lb. per foot, and the thrust on the rod is 18 tons. Estimate the maximum compressive stress in the rod. $E \doteq 30 \times 10^6$ lb./sq. in. *Ans.*, 14,092 lb./sq. in.

CHAPTER XIII

THICK CYLINDERS AND SPHERES

127. Thick Cylindrical Shells. The equations which follow are built up on the assumptions that the material is isotropic, and that a transverse section of the cylinder, which is a plane before being subjected to pressure, remains a plane after being subjected to pressure. The latter assumption is approximately true at a fair distance from the ends of the cylinder ; it implies that the strain, in a longitudinal direction, is constant or zero.

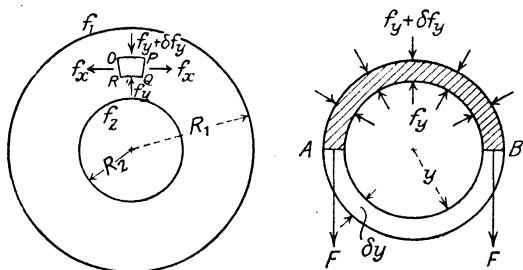



FIG. 97

Consider a cylinder of outer and inner radii R_1 and R_2 (Fig. 97), at which the corresponding fluid pressures are f_1 and f_2. An element such as *OPQR* of thickness δy at radius y is subjected to radial compressive stresses of magnitude f_y and $f_y + \delta f_y$ on the faces *QR* and *OP* respectively, and on the faces *OR* and *PQ* a circumferential or hoop stress of magnitude f_x.

Considering the equilibrium of a thin cylindrical element embracing *OPQR*, and of unit axial length, the resultant force tending to burst the cylinder at *AB* is given by

$$2f_y y - 2(f_y + \delta f_y)(y + \delta y)$$

and this is resisted by the force $2F$

$$= 2f_x \delta y$$

$$\therefore\ 2f_y y - 2(f_y + \delta f_y)(y + \delta y) = 2f_x \delta y$$

Neglecting products of small quantities, this reduces to

$$f_x \delta y = -f_y \delta y - y \delta f_y$$

or
$$f_x + y\frac{\delta f_y}{\delta y} + f_y = 0$$

which, when δy is reduced indefinitely, reduces to

$$f_x + y\frac{df_y}{dy} + f_y = 0$$

or
$$\frac{d}{dy}(yf_y) + f_x = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Let e_z and f_z be the strain and stress respectively in an axial direction, then, since a transverse section remains a plane,

$$e_z = \frac{1}{E}\left(f_z - \frac{f_x - f_y}{m}\right) \quad . \quad . \quad . \quad . \quad (2)$$

Assuming that the stress in an axial direction is uniform over the cross-section, hence from (2), $(f_x - f_y)$ must be constant.

Let
$$f_x - f_y = 2a \quad . \quad . \quad . \quad (3)$$

Then from (1)

$$\frac{d}{dy}(yf_y) + f_y + 2a = 0$$

and
$$y\frac{df_y}{dy} + 2f_y + 2a = 0$$

$\therefore$
$$\frac{df_y}{f_y + a} = -2\frac{dy}{y}$$

and integrating each side

$$\log_e(f_y + a) = -\log_e y^2 + \text{a constant}$$

or
$$\log_e\{(f_y + a) \times y^2\} = \text{a constant}$$

Let
$$(f_y + a)y^2 = b$$

Then
$$f_y = \frac{b}{y^2} - a \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from (3)
$$f_x = \frac{b}{y^2} + a \quad . \quad . \quad . \quad . \quad . \quad (5)$$

At R_1, $f_y = f_1$, and at R_2, $f_y = f$

hence $f_1 = \dfrac{b}{R_1^2} - a$ and $f_2 = \dfrac{b}{R_2^2} - a$

from which
$$a = \frac{f_2 R_2^2 - f_1 R_1^2}{R_1^2 - R_2^2} \quad . \quad . \quad . \quad . \quad (6)$$

and $$b = (f_2 - f_1)\frac{R_1^2R_2^2}{R_1^2 - R_2^2} \quad . \quad . \quad . \quad (7)$$

From (4) and (5)

$$f_y = \frac{1}{R_1^2 - R_2^2}\left\{(f_2 - f_1)\frac{R_1^2R_2^2}{y^2} - (f_2R_2^2 - f_1R_1^2)\right\} \quad . \quad (8)$$

$$\text{and } f_x = \frac{1}{R_1^2 - R_2^2}\left\{(f_2 - f_1)\frac{R_1^2R_2^2}{y^2} + (f_2R_2^2 - f_1R_1^2)\right\} \quad . \quad (9)$$

128. **Zero External Pressure.** In the case of hydraulic cylinders $f_1 = 0$, hence

$$f_y = \frac{1}{R_1^2 - R_2^2}\left\{f_2\frac{R_1^2R_2^2}{y^2} - f_2R_2^2\right\}$$

$$= \frac{f_2R_2^2}{R_1^2 - R_2^2}\left\{\frac{R_1^2}{y^2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\text{and } f_x = \frac{f_2R_2^2}{R_1^2 - R_2^2}\left\{\frac{R_1^2}{y^2} + 1\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

f_x and f_y are each maximum where $y = R_2$.

∴ The maximum value of radial compressive stress is

$$f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the maximum value of tensile hoop stress

$$= f_2\frac{R_1^2 + R_2^2}{R_1^2 - R_2^2} \quad . \quad . \quad . \quad . \quad (13)$$

$$= f_2\frac{k^2 + 1}{k^2 - 1} \text{ where } k = \frac{R_1}{R_2}$$

129. **Application of Compound Stress Theories.** Let f be the yield stress of the material in tension.

(*a*) *Maximum Principal Stress.*

According to this theory, the greatest value of the tensile hoop stress $f_x = f$, and the greatest allowable value of f_2 is given by

$$f = f_2\frac{k^2 + 1}{k^2 - 1}$$

or $$f_2 = \frac{k^2 - 1}{k^2 + 1}f \quad . \quad . \quad . \quad . \quad (1)$$

(*b*) *Maximum Principal Strain.*

The maximum principal strain will occur at R_2 and is given by

$$e_{max} = \frac{1}{E}\left\{f_x + \frac{f_y}{m}\right\} \text{ since } f_y \text{ is compressive.}$$

The equivalent stress $= f_x + \dfrac{f_y}{m}$

$$= \frac{f_2 R_2^2}{R_1^2 - R_2^2}\left\{\frac{R_1^2}{R_2^2}\left(1 + \frac{1}{m}\right) + \left(1 - \frac{1}{m}\right)\right\}$$

$$= \frac{f_2}{k^2 - 1}\left\{\frac{k^2(m + 1) + (m - 1)}{m}\right\}$$

According to this theory the greatest value of f_2 is obtained when the above equivalent stress is equal to f or

$$f_2 = \frac{m\,(k^2 - 1)f}{k^2(m + 1) + (m - 1)} \quad . \quad . \quad (2)$$

(*c*) *Maximum Shear Stress, or Stress-difference, Theory.*

According to this theory, we have, for values of f_x and f_y at radius R_2,

$$\frac{f}{2} = \frac{f_x + f_y}{2}$$

hence

$$f = 2f_2\frac{R_2^2}{R_1^2 - R_2^2}\cdot\frac{R_1^2}{R_2^2}$$

$$= 2f_2\frac{k^2}{k^2 - 1}$$

$$\therefore \quad f_2 = \frac{k^2 - 1}{2k^2}f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(*d*) *Haigh's Strain Energy Theory.*

From par. 28 (1), the strain energy is given by

$$\left\{f_x^2 + f_y^2 + \frac{2f_x f_y}{m}\right\}\frac{1}{2E}$$

and according to this theory is equal to $\dfrac{f^2}{2E}$.

$$\therefore \quad f^2 = f_x^2 + f_y^2 + \frac{2f_x f_y}{m}$$

$$= \left(\frac{f_2}{k^2-1}\right)^2 \left\{(k^2-1)^2 + (k^2+1)^2 + \frac{2(k^2-1)(k^2+1)}{m}\right\}$$

$$= \left(\frac{f_2}{k^2-1}\right)^2 \left\{2\,\frac{m+1}{m}\,k^4 + 2\,\frac{m-1}{m}\right\}$$

$$\therefore \quad f_2 = \frac{(k^2-1)f}{\sqrt{2\left\{\frac{m+1}{m}\,k^4 + \frac{m-1}{m}\right\}}} \quad . \quad . \quad . \quad (4)$$

EXAMPLE 1.

The cylinder of a hydraulic press has an internal diameter of 10 in., and the water pressure is 1,400 lb./sq. in. Find a suitable thickness for the cylinder if the maximum stress is not to exceed 3,000 lb./sq. in. Sketch a diagram showing the distribution of hoop stress through the wall of the cylinder. (Lond. Univ., 1918.)

From par. 129 (1),

$$f_2 = f \cdot \frac{k^2 - 1}{k^2 + 1}$$

or

$$\frac{k^2 - 1}{k^2 + 1} = \frac{1400}{3000} = \frac{7}{15}$$

and

$$k^2 = \frac{11}{4} = \left(\frac{R_1}{R_2}\right)^2$$

$$\therefore \quad \frac{R_1}{R_2} = \sqrt{\frac{11}{4}} = 1{\cdot}66$$

$$R_1 = 1{\cdot}66R_2 = 1{\cdot}66 \times 5$$

$$= 8{\cdot}3$$

$$\text{Thickness of cylinder} = 8{\cdot}3 - 5$$

$$= 3{\cdot}3 \text{ in.}$$

$$f_x = \frac{f_2 R_2^2}{R_1^2 - R_2^2}\left\{\frac{R_1^2}{y^2} + 1\right\} = \frac{1400 \times 25}{(8{\cdot}3)^2 - (5)^2}\left\{\frac{(8{\cdot}3)^2}{y^2} + 1\right\}$$

$$= 795\left\{\frac{69}{y^2} + 1\right\}$$

y in.	5	6	7	8	8·3
f_x lb./sq. in. .	3000	2320	1910	1660	1590

In this example the value of the compressive radial stress is given by

$$f_y = 795 \left\{ \frac{69}{y^2} - 1 \right\}$$

y in.	5	6	7	8	8·3
f_y lb./sq. in. .	1400	750	320	63·6	0

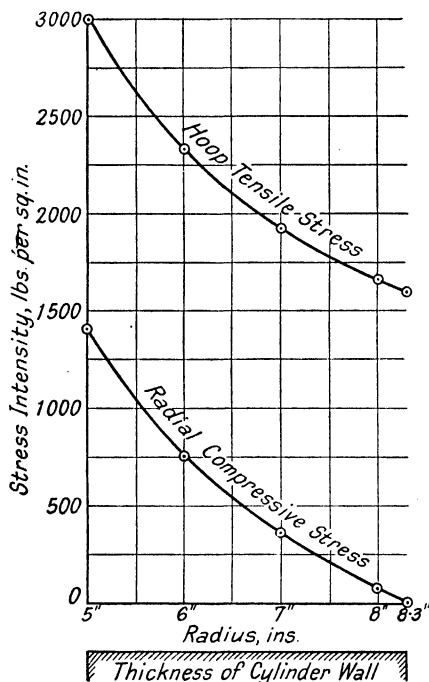



FIG. 98

The values of f_x and f_y are shown plotted against the corresponding values of y in Fig. 98.

130. **Compound Cylinders.** From Fig. 98 it will be observed that there is a great variation in the intensity of the stress in the wall of a thick cylinder subjected to internal fluid pressure, and thus the material is not used to the best advantage. In order to secure a more uniform stress distribution, one method is to build up the cylinder by shrinking one tube on

the outside of another. The inner tube is brought to a state of compression by the contraction, on cooling, of the external tube; this tube will therefore be under a similar internal pressure, and hence be in a state of tension. When the compound tube is subjected to internal fluid pressure, the resultant stress is the algebraic sum of that due to the shrinking and that due to the internal pressure. The resultant tensile stress at the inner surface of the inner tube is not so large as if the cylinder were composed of one thick tube, and the final tensile stress at the inner surface of the outer tube is larger than if the cylinder were composed of one thick tube; thus a more even stress distribution is obtained.

Equations similar to those given in pars. 127 and 128 may be used to calculate the stress due to shrinking, due attention being paid to the sign of f_x and f_y.

In the case of gun-making, it is not an easy matter to turn and bore long tubes to the degree of accuracy required for shrinking. It is usual in this case to wind the inner tube with ribbon of a rectangular section with sufficient tension to bring the tube to a state of compression.

EXAMPLE 2.

A compound tube is composed of a tube, 10 in. internal diameter and 1 in. thick, shrunk unto a tube 10 in. external diameter and 1 in. thick. The radial pressure at the junction is 1,100 lb./sq. in. The compound tube is subjected to an internal fluid pressure of 10,000 lb./sq. in. Show by a diagram how the hoop stress varies over the wall of the compound tube, and compare it with that over a solid tube of 12 in. external diameter and 2 in. thick under the same internal pressure.

Using equations (4) and (5) of par. 127, and considering shrinking only.

Outer tube. $$f_y = \frac{b}{y^2} - a$$

$f_y = 0$ at $y = 6$, and $f_y = 1100$ at $y = 5$.

hence $b - 36a = 0$

and $b - 25a = 27{,}500$

$\therefore \quad a = 2500$ and $b = 90{,}000$.

Inner tube.

$f_y = 0$ at $y = 4$, and $f_y = 1100$ at $y = 5$,

hence $b_1 - 16a_1 = 0$

and $b_1 - 25a_1 = 27{,}500$

$$\therefore \quad a_1 = -\frac{27,500}{9} \text{ and } b_1 = -\frac{440,000}{9}$$

Outer tube.

$$f_x = \frac{b}{y^2} + a = \frac{90,000}{y^2} + 2500$$

at $y = 6, f_x = \dfrac{90,000}{36} + 2500 = 5000$ lb./sq. in. tension

at $y = 5, f_x = \dfrac{90,000}{25} + 2500 = 6100$ lb./sq. in. tension

Inner tube. $f_x = -\dfrac{440,000}{9y^2} - \dfrac{27,500}{9}$

$$= -\frac{49,000}{y^2} - 3055$$

at $y = 5, f_x = -\dfrac{49,000}{25} - 3055 = -1950 - 3055$

$= -5005$ lb./sq. in. or 5005 lb./sq. in. compression

at $y = 4, f_x = -\dfrac{49,000}{16} - 3055 = -3055 - 3055$

$= -6110$ lb./sq. in. or 6110 lb./sq. in. compression.

Considering internal pressure only,
at $y = 6, f_y = 0$, and at $y = 4, f_y = 10,000$,

hence $\qquad b_2 - 36a_2 = 0$

and $\qquad b_2 - 16a_2 = 160,000$

$\therefore \qquad a_2 = 8000$, and $b_2 = 288,000$

at $y = 4, f_x = \dfrac{288,000}{16} + 8000 = 26,000$ lb./sq. in. tension

at $y = 5, f_x = \dfrac{288,000}{25} + 8000 = 19,500$ lb./sq. in. tension

at $y = 6, f_x = \dfrac{288,000}{36} + 8,000 = 16,000$ lb./sq. in. tension

Resultant stresses.

Outer tube.

At $y = 6, f_x = 16{,}000 + 5000 = 21{,}000$ lb./sq. in. tension
at $y = 5, f_x = 19{,}500 + 6100 = 25{,}600$ lb./sq. in. tension

Inner tube.

At $y = 5, f_x = 19{,}500 - 5005 = 14{,}495$ lb./sq. in. tension
at $y = 4, f_x = 26{,}000 - 6110 = 19{,}890$ lb./sq. in. tension

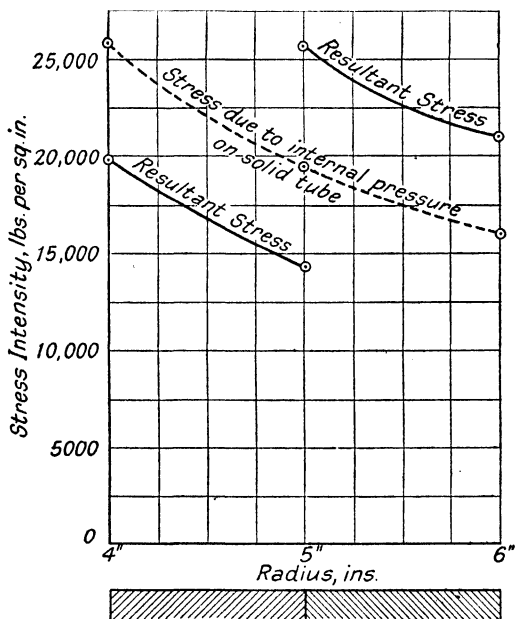



Fig. 99

The stress diagrams are shown by Fig. 99. It will be observed that the distribution of stress is more uniform in the compound tube than in the solid tube. The greater the number of tubes, the more uniform is the stress distribution.

131. **Thick Spherical Shell.** Let R_1 and R_2 be the outer and inner radii of the shell, f_1 and f_2 being the corresponding pressures, and let f_x be the tensile stress in a circumferential direction, and f_y the compressive stress in a radial direction at radius y. At this radius suppose an element of thickness δy to be displaced in a radial direction by the amount μ.

Circumferential strain at radius y

$$= \frac{2\pi(y + \mu) - 2\pi y}{2\pi y}$$

$$= \frac{\mu}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a radial direction the width of the element after strain

$$= y + \delta y + \mu + \delta\mu - (y + \mu)$$

$$= \delta y + \delta\mu$$

Then radial strain $= \dfrac{\delta y + \delta\mu - \delta y}{\delta y} = \dfrac{\delta\mu}{\delta y}$

$$= \frac{d\mu}{dy} \text{ in the limit} \quad . \quad . \quad . \quad (2)$$

but $$\frac{d\mu}{dy} = \frac{1}{E}\left(-f_y - \frac{2f_x}{m}\right) \quad . \quad . \quad . \quad . \quad (3)$$

and $$\frac{\mu}{y} = \frac{1}{E}\left(f_x - \frac{f_x}{m} + \frac{f_y}{m}\right) \quad . \quad . \quad . \quad (4)$$

Now consider the forces acting on an elementary shell of thickness δy at radius y, whose internal pressure is f_y and external pressure $f_y + \delta f_y$.

Across a diametral plane the bursting force is given by

$$\pi y^2 f_y - \pi(y + \delta y)^2 (f_y + \delta f_y)$$

and the resisting force $= 2\pi y \,.\, \delta y f_x$

hence $$2\pi y \delta y f_x = \pi y^2 f_y - \pi(y + \delta y)^2 (f_y + \delta f_y)$$

and $$f_x = -f_y - \frac{y}{2}\frac{dfy}{dy} \quad . \quad . \quad . \quad . \quad (5)$$

$\therefore$ $$\frac{df_x}{dy} = -\frac{df_y}{dy} - \frac{y}{2}\frac{d^2 f_y}{dy^2} - \frac{1}{2}\frac{df_y}{dy} \quad . \quad . \quad (6)$$

Multiplying equation (4) by y, differentiating and subtracting from (3), we have

$$(m - 1)\, y\frac{df_x}{dy} + y\frac{df_y}{dy} + (m + 1)(f_y + f_x) = 0 \quad . \quad (7)$$

Substituting the values of f_x and $\frac{df_x}{dy}$ from (5) and (6)

$$\frac{d^2f_v}{dy^2} + \frac{4}{y}\frac{df_v}{dy} = 0$$

hence
$$y^4 \frac{df_v}{dy} = \text{a constant} = X$$

then
$$\frac{df_v}{dy} = \frac{X}{y^4}$$

$\therefore$
$$f_v = -\frac{X}{3y^3} + Y \quad . \quad . \quad . \quad . \quad (8)$$

also from (5)
$$f_x = -\frac{X}{6y^3} - Y \quad . \quad . \quad . \quad . \quad (9)$$

At $y = R_1$, $f_v = f_1$, and at $y = R_2$, $f_v = f_2$.

$$f_1 = -\frac{X}{3R_1^3} + Y$$

$$f_2 = -\frac{X}{3R_2^3} + Y$$

Solving, we get

$$X = \frac{3(f_2 - f_1)R_1^3R_2^3}{R_2^3 - R_1^3} \quad . \quad . \quad . \quad (10)$$

$$Y = \frac{f_2R_2^3 - f_1R_1^3}{R_2^3 - R_1^3} \quad . \quad . \quad . \quad . \quad (11)$$

$\therefore$
$$f_v = -\frac{(f_2 - f_1)R_1^3R_2^3}{(R_2^3 - R_1^3)y^3} + \frac{f_2R_2^3 - f_1R_1^3}{R_2^3 - R_1^3} \quad . \quad . \quad (12)$$

$$f_x = -\frac{(f_2 - f_1)R_1^3R_2^3}{2y^3(R_2^3 - R_1^3)} - \frac{f_2R_2^3 - f_1R_1^3}{R_2^3 - R_1^3} \quad . \quad . \quad (13)$$

In the special case, when $f_1 = 0$

$$f_v = \frac{f_2R_2^3}{R_1^3 - R_2^3}\left\{\frac{R_1^3}{y^3} - 1\right\} \quad . \quad . \quad . \quad . \quad (14)$$

$$f_x = \frac{f_2R_2^3}{R_1^3 - R_2^3}\left\{\frac{R_1^3}{2y^3} + 1\right\} \quad . \quad . \quad . \quad . \quad (15)$$

The maximum stress is given by

$$f_{x_{max}} = \frac{f_2}{2}\left\{\frac{R_1^3 + 2R_2^3}{R_1^3 - R_2^3}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

which occurs at $y = R_2$.

EXAMPLES XIII

1. Deduce expressions for the intensities of the radial and circumferential stresses at any point in the cross-section of a thick-walled cylinder subjected to internal pressure. Find the circumferential stress at the inner and outer radius respectively in the case of a pipe, 4 in. internal diameter and 1½ in. thick when subjected to an internal pressure of 1,000 lb./sq. in. (Lond. Univ., 1919.) *Ans.*, 1,980 and 962 lb./sq. in. respectively.

2. The cylinder of a hydraulic ram is 10 in. internal diameter and 2 in. thick. Find the maximum hoop stress in the material and the radial and hoop stresses at a point in the barrel ¾ in. from the inner surface when the fluid pressure inside the cylinder is 1,300 lb./sq. in. (Lond. Univ., 1925.) *Ans.*, 4,000, 650, and 3,350 lb./sq. in. respectively.

3. A cylinder 10 in. internal diameter has to sustain an internal pressure of 1,250 lb./sq. in. What thickness must the walls be if the stress in the material is not to exceed 4,000 lb./sq. in., and what will be the tension at the outside of the cylinder walls ? Prove any formula you may use for this thick cylinder. (A.M.I.Mech.E., 1926.) *Ans.*, 1·91 in., 2,760 lb./sq. in.

4. Determine the radial and hoop stresses at all points in a thick-walled cylinder of internal and external radii 3 in. and 6 in. respectively when acted upon by internal and external pressures of 8 tons and 3 tons per square inch respectively. Show by a diagram how each stress varies over the thickness.

5. A cylinder of 6 in. internal diameter and 8 in. external diameter is subjected to liquid pressure from inside. There is also a compression load of 50 tons applied at the ends of the cylinder. Find the greatest pressure of the liquid so that the maximum stress in the material may not exceed 4 tons/sq. in. (Lond. Univ., 1919.) *Ans.*, 2,520 lb./sq. in.

6. A compound tube is made by shrinking one tube on another, the final dimensions being : External diameter 10 in., internal diameter 6 in., diameter at junction of tubes 9 in. If the radial pressure on the external surface of the inner tube at the common 4·5 in. radius is 4,000 lb./sq. in., determine the maximum tensile stress in the metal of the outer tube. (Lond. Univ., 1916.) *Ans.*, 38,000 lb./sq. in.

7. A spherical shell is 12 in. external diameter and 2 in. thick. What is the tensile stress at the inner and outer surface when the internal pressure is 1,000 lb./sq. in. ? *Ans.*, 1,135 and 632 lb./sq. in. respectively.

8. Indicate briefly the derivation of Lamé's equations for the stresses in thick tubes subjected to external or internal pressure.

A bronze liner, 12 in. external diameter and 4 ft. long is shrunk on a steel shaft 6 in. diameter, the shrinkage allowance being such that the maximum principal stress in the liner is 5 tons/sq. in. Estimate the torque which can be applied to the liner before it slips on the shaft, if the coefficient of friction is 0·25. (Lond. Univ., 1936.) *Ans.*, 2,040 ton in.

CHAPTER XIV

ROTATION OF RINGS AND DISCS—WHIRLING OF SHAFTS

132. **Rotating Ring.** A thin ring, such as that shown by Fig. 100, when rotating about an axis through its centre of gravity, has induced in it a hoop stress which is almost uniform when the thickness of the ring is small compared with its diameter.

Let w = the weight of unit volume in pounds.

r = the mean radius of the ring in feet.

a = the area of the cross-section of the ring in sq. ft.

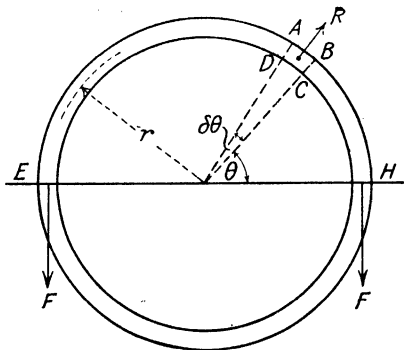



Fig. 100

The centrifugal force due to the element subtended by small angle $\delta\theta$

$$= \frac{w \times a \times r\delta\theta}{g} \omega^2 r$$

where ω = the angular velocity of rotation in radian/sec.

The vertical component of this centrifugal force

$$= \frac{war\delta\theta}{g} \omega^2 r \sin\theta$$

The total force tending to burst the ring across the diameter EH

$$= \int_0^\pi \frac{war}{g} \omega^2 r \sin \theta d\theta$$

$$= 2 \frac{war}{g} \omega^2 r$$

The resisting force $= 2F$

or $= 2af$ where f is the hoop stress lb./sq. ft.

$$\therefore \quad 2af = \frac{2war}{g} \omega^2 r$$

$$f = \frac{w}{g} \omega^2 r^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$f = \frac{w}{g} v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v = the linear velocity at radius r in ft./sec.

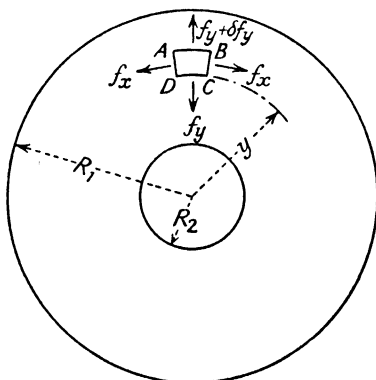



FIG. 101

133. **Flat Rotating Disc of Uniform Thickness.** Let the inner and outer radii be R_2 and R_1 respectively, and t be the thickness of the disc. An element of the disc $ABCD$ (Fig. 101) at radius y is acted on by stresses f_y and $f_y + \delta f_y$ on the faces CD and AB respectively, and by stresses f_x on the faces AD and CB. On the flat faces of the disc there is no normal stress, and hence there is free strain in the direction of the axis.

The radial force on the element, due to rotation,

$$= \frac{w}{g}\,\omega^2 y t y \delta\theta \delta y$$

where $\delta\theta$ is the angle between AD and BC
and is equal to

$$\left\{ f_y y\delta\theta + 2f_x \delta y \sin\frac{\delta\theta}{2} - (f_y + \delta f_y)(y + \delta y)\delta\theta \right\} t$$

$$\doteqdot \left\{ f_y y\delta\theta + 2f_x \delta y \frac{\delta\theta}{2} - (f_y + \delta f_y)(y + \delta y)\delta\theta \right\} t$$

and neglecting the product of small quantities

$$f_x = f_y + y\frac{df_y}{dy} + \frac{w}{g}\,\omega^2 y^2 \quad . \quad . \quad . \quad . \quad (1)$$

If, owing to strain, y increases to $y + \mu$ we have, as in par. 131,

$$\text{circumferential strain} = \frac{\mu}{y} = \frac{1}{E}\left(f_x - \frac{f_y}{m}\right) \quad . \quad . \quad (2)$$

and

$$\text{radial strain} = \frac{d\mu}{dy} = \frac{1}{E}\left(f_y - \frac{fx}{m}\right) \quad . \quad . \quad (3)$$

From (2) $\dfrac{d\mu}{dy} = \dfrac{y}{E}\left(\dfrac{df_x}{dy} - \dfrac{1}{m}\dfrac{df_y}{dy}\right) + \dfrac{1}{E}\left(f_x - \dfrac{f_y}{m}\right)$

and substituting in (3)

$$y\frac{df_x}{dy} - f_y\left(\frac{1}{m} + 1\right) + f_x\left(\frac{1}{m} + 1\right) - \frac{y}{m}\frac{df_y}{d_y} = 0$$

$$f_x - f_y + \frac{m}{m+1}\,y\frac{df_x}{dy} - \frac{1}{m+1}\,y\frac{df_y}{dy} = 0$$

and substituting the value of $f_x - f_y$ from (1)

$$y\frac{df_y}{dy} + \frac{w}{g}\,\omega^2 y^2 + \frac{m}{m+1}\,y\frac{df_x}{dy} - \frac{1}{m+1}\,y\frac{df_y}{dy} = 0$$

$$\frac{m}{m+1}\left\{ y\frac{df_x}{dy} + y\frac{df_y}{dy} \right\} + \frac{w}{g}\,\omega^2 y^2 = 0$$

$$\frac{d}{dy}\{f_x + f_y\} + \frac{m+1}{m}\,\frac{w}{g}\,\omega^2 y = 0$$

or $\quad f_x + f_y = X - \dfrac{m+1}{2m}\,\dfrac{w}{g}\,\omega^2 y^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$

Substituting the value of f_x in (1)

$$2f_y + y\frac{df_y}{dy} + \frac{w}{g}\omega^2 y^2 = X - \frac{m+1}{2m}\frac{w}{g}\omega^2 y^2$$

or
$$2f_y + y\frac{df_y}{dy} = X - \frac{3m+1}{2m}\frac{w}{g}\omega^2 y^2$$

or, on multiplying each side by y,

$$2f_y y + y^2\frac{df_y}{dy} = Xy - \frac{3m+1}{2m}\frac{w}{g}\omega^2 y^3$$

or
$$\frac{d}{dy}(y^2 f_y) = Xy - \frac{3m+1}{2m}\frac{w}{g}\omega^2 y^3$$

$$\therefore \quad y^2 f_y = \frac{Xy^2}{2} - \frac{3m+1}{8m}\frac{w}{g}\omega^2 y^4 + Y$$

$$\therefore \quad f_y = \frac{X}{2} + \frac{Y}{y^2} - \frac{3m+1}{8m}\frac{u}{g}\omega^2 y^2 \quad . \quad . \quad (5)$$

and from (4)

$$f_x = \frac{X}{2} - \frac{Y}{y^2} - \frac{m+3}{8m}\frac{w}{g}\omega^2 y^2 \quad . \quad . \quad (6)$$

134. **Solid Disc.** At $y = R_1, f_y = 0$, and at $y = 0$, $\mu = 0$, the latter condition gives $Y = 0$, and from (5)

$$0 = \frac{X}{2} - \frac{3m+1}{8m}\frac{w}{g}\omega^2 R_1^2$$

$$\therefore X = \frac{3m+1}{4m}\frac{w}{g}\omega^2 R_1^2$$

$$\therefore f_y = \frac{3m+1}{8m}\{R_1^2 - y^2\}\frac{w\,\omega^2}{g} \quad . \quad . \quad . \quad . \quad (7)$$

$$f_x = \frac{1}{8m}\{(3m+1)R_1^2 - (m+3)y^2\}\frac{w\omega^2}{g} \quad . \quad . \quad (8)$$

The maximum stress occurs at the centre of rotation where $y = 0$ and

$$f_x = f_y = \frac{3m+1}{8m}R_1^2\frac{w}{g}\omega^2 \quad . \quad . \quad . \quad . \quad (9)$$

135. **Disc with Central Hole.** In order to determine the constants X and Y, in the equations (5) and (6), for this case we have at $y = R_1, f_y = 0$, and at $y = R_2, f_y = 0$.

Hence $$X = \frac{3m+1}{4m}\frac{w}{g}\omega^2(R_1^2 + R_2^2) \quad . \quad . \quad . \quad . \quad (10)$$

and $$Y = -\frac{3m+1}{8m}\frac{w}{g}\omega^2 R_1^2 R_2^2 \quad . \quad . \quad . \quad . \quad (11)$$

from which

$$f_y = \frac{3m+1}{8m}\left\{R_1^2 + R_2^2 - \frac{R_1^2R_2^2}{y^2} - y^2\right\}\frac{w}{g}\omega^2 \quad . \quad . \quad (12)$$

$$f_x = \frac{3m+1}{8m}\left\{R_1^2 + R_2^2 + \frac{R_1^2R_2^2}{y^2} - \frac{m+3}{3m+1}y^2\right\}\frac{w}{g}\omega^2 \quad (13)$$

The maximum value of the hoop stress f_x is obtained when $y = R_2$, and is given by

$$f_{x\ max} = \frac{3m+1}{4m}\left\{R_1^2 + \frac{(m-1)}{3m+1}R_2^2\right\}\frac{w}{g}\omega^2. \quad . \quad . \quad (14)$$

the greatest value of f_y is found when $\frac{df_y}{dy} = 0$

or $$\frac{2R_1^2R_2^2}{y^3} - 2y = 0$$

which gives $$y = \sqrt{R_1R_2}$$

and $$f_{y\ max} = \frac{3m+1}{8m}(R_1 - R_2)^2\frac{w}{g}\omega^2 \quad . \quad . \quad . \quad (15)$$

When R_2 is so small that R_2^2 is negligible, then the greatest value of the hoop stress reduces to

$$\frac{3m+1}{4m}R_1^2\frac{w}{g}\omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Comparing (16) with (9), it will be observed that the maximum stress in a rotating disc is twice as large, when there is a small hole at its axis of rotation, as that when the disc is solid.

136. Disc of Uniform Strength. If the disc is to have uniform strength the hoop stress and the radial stress must be equal to one another at all points, and have the same value

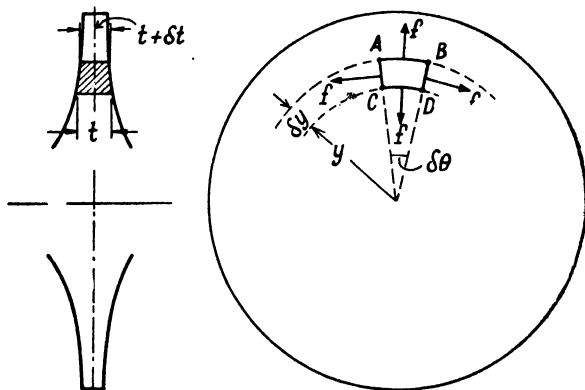



FIG. 101A

throughout the disc. Hence $f_x = f_y = f$ and is a constant for the disc.

Consider the equilibrium of the element *ABCD*.

Outward radial force acting on face at *AB*

$$= f(t + \delta t)\,(y + \delta y)\delta\theta$$
$$\fallingdotseq f(ty + y\delta t + t\delta y)\delta\theta$$

Centrifugal force acting on the element

$$= \frac{w \text{ volume}}{g}\,\omega^2 y$$
$$\fallingdotseq \frac{w}{g}\,y \,.\, \delta\theta \,.\, \delta y \,.\, t\omega^2 y$$
$$= \frac{wy^2\omega^2 t\delta y\delta\theta}{g}$$

Total outward radial force acting on the element

$$= f(ty + y\delta t + t\delta y)\delta\theta + \frac{wy^2\omega^2 t\delta y\delta\theta}{g}$$

Radial inward force acting on face at $CD = fty\delta\theta$. Inward radial force due to components of forces acting on faces at AC and $BD \fallingdotseq 2ft\delta y\frac{\delta\theta}{2} = ft\delta y\delta\theta$. Total inward radial force $= fty\delta\theta + ft\delta y \,.\, \delta\theta$.

For equilibrium the resultant radial force on the element is zero.

$$\therefore \quad f(ty + y\delta t + t\delta y)\delta\theta + \frac{wy^2\omega^2 t\delta y\delta\theta}{g}$$
$$= fty\delta\theta + ft\delta y \,.\, \delta\theta$$

hence
$$f \,.\, y \,.\, \delta t \,.\, \delta\theta + \frac{w \,.\, y^2 \,.\, \omega^2 \,.\, t\delta y \,.\, \delta\theta}{g} = 0$$

or
$$\frac{dt}{t} = -\frac{w\omega^2}{gf} y\, dy$$

and
$$\log_e t = -\frac{w\omega^2}{gf}\frac{y^2}{2} + c.$$

$$\therefore \quad t = Ae^{-\frac{w\omega^2}{2gf}y^2}$$

where A is an arbitrary constant of integration.

When $y = 0$, i.e. at axis of disc let thickness be t_o

then
$$t_o = A$$

Hence
$$t = t_o e^{-\frac{w\omega^2}{2gf}y^2}$$

The rotor of a De Laval steam turbine is an example of a solid disc of uniform strength.

EXAMPLE 1.

A uniform steel disc 24 in. in diameter runs at 3,000 r.p.m. Show by a diagram how the radial and hoop stresses vary in the disc. What is the magnitude of the greatest stress in a similar disc, which has a hole $\frac{1}{16}$ in. through it at the axis? Assume 1 cub. in. of steel to weigh 0·29 lb. and $E = 30 \times 10^6$ lb./sq.in.

$$f_y = \frac{3m+1}{8m}\left\{R_1^2 - y^2\right\}\frac{w\,\omega^2}{g}$$
$$= \frac{13}{32}\left\{144 - y^2\right\}\frac{0{\cdot}29}{12 \times 32} \times \left(\frac{3000}{60} \times 2\pi\right)^2$$
$$= 4420 - 30{\cdot}63\, y^2 \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$f_x = \frac{1}{8m}\left\{(3m+1)R_1^2 - (m+3)y^2\right\}\frac{w}{g}\omega^2$$
$$= 4420 - 16{\cdot}45 y^2 \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

y (in.)	0	3	6	9	12
f_y (lb./sq. in.)	4420	4145	3320	1940	0
f_x (lb./sq. in.)	4420	4272	3828	3090	2050

The stresses are shown plotted to a base of radius in Fig. 102.

Maximum stress in disc with small axial hole is given by par. 135 (16)

$$= \frac{3m+1}{4m} \times 144 \times \frac{0{\cdot}28}{12 \times 32} \times \left(\frac{3000}{60} \times 2\pi\right)^2$$

$$= 8840 \text{ lb./sq. in.}$$

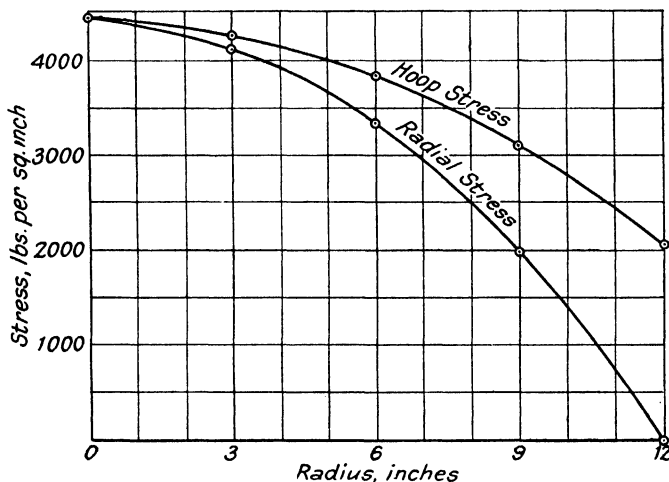



FIG. 102

EXAMPLE 2.

A De Laval steam turbine rotor is 6 in. diameter below the blade ring, and ¼ in. thick, the running speed being 30,000 r.p.m. If the material weighs 0·28 lb./cub. in., and the allowable stress is 20,000 lb./sq. in., what is the thickness of the rotor at a radius of 1·5 in. and at the centre? Assume uniform strength.

$$t = t_0 e^{-\frac{w}{2g}\frac{\omega^2 y^2}{f}}$$

$$t = t_0 e^{-\frac{0{\cdot}28}{24 \times 32} \times \frac{250{,}000 \times 4\pi^2}{20{,}000} y}$$

$$= t_0 e^{-0{\cdot}18 y^2}$$

$$0{\cdot}25 = t_0 e^{-0{\cdot}18 \times 9}$$

$$= t_0 e^{-1{\cdot}62}$$

$$= 0{\cdot}1979 t_0$$

$$\therefore \qquad t_o = \frac{0 \cdot 25}{0 \cdot 1979}$$

$$= 1 \cdot 263 \text{ in.} = \text{thickness at centre}$$

$$t = 1 \cdot 263 \times e^{-0 \cdot 18 \times (1 \cdot 5)}$$

$$= 1 \cdot 263 \times e^{-0 \cdot 405}$$

$$= 0 \cdot 8436 \text{ in.} = \text{thickness at } 1\tfrac{1}{2} \text{ in. radius}$$

137. **Whirling of Shafts.** The geometrical axis of a shaft does not usually coincide with the axis of rotation, on account of the dead weight of the shaft and its loads, or else by reason of initial crookedness. Owing to rotation, centrifugal forces are set up, and these tend to cause a still greater difference between the axes.

The centrifugal forces are balanced by the forces arising from the stiffness of the shaft. If the speed of rotation be such that the centrifugal forces are able to overcome the forces due to stiffness, then the deflection and stress will increase until fracture occurs. The speed at which this unstable condition arises is called the *whirling speed* of the shaft. Whirling also takes place at speeds higher than this fundamental speed, but the latter is usually of most importance. By considering the possible arrangements of nodes between the bearings it will be clear from what follows that the second, third, fourth, etc., critical speeds are to the first as 1 : 4 : 9 : 16, etc.

138. **Concentrated Load.** If the shaft carries a load W lb. such that the weight of the shaft is negligible, and its dimensions small in comparison to its length, the critical speed can be found by equating the elastic forces to the centrifugal forces.

Let y be the deflection of the shaft at the load (inches), and ω the critical angular velocity of the shaft (radian/second). Let a and b be the length into which W divides the shaft (inches).

$$\text{Centrifugal force} = \frac{W}{g}\,\omega^2 y$$

where g is the acceleration due to gravity (inches/sec./sec.).

If W_1 is the static load which will cause y, then from (6) par. 79.

$$y = \frac{W_1 a^2 b^2}{3(a+b)EI}$$

The stiffness of the shaft or the load per unit deflection $= \frac{W_1}{y} = \frac{3(a+b)EI}{a^2b^2} = k.$

Up to the whirling speed the elastic restoring forces are proportional to the deflection.

Hence, restoring force $= ky$.

$$\therefore \qquad ky = \frac{W}{g}\omega^2 y$$

$$\omega = \sqrt{\frac{kg}{W}}$$

or

$$\omega = \sqrt{\frac{3(a+b)EIg}{Wa^2b^2}} \qquad . \quad . \quad . \quad (1)$$

If there is an initial deflection h (inches) before rotation, and y_1 is the added deflection due to rotation at the whirling speed,

$$\text{Centrifugal force} = \frac{W}{g}\omega^2(h + y_1)$$

and

$$\frac{W}{g}\omega^2(h + y_1) = ky_1$$

$$\therefore \qquad y_1 = -\frac{W\omega^2 h}{W\omega^2 - kg} = \frac{\omega^2 h}{\frac{kg}{W} - \omega^2} \qquad . \quad . \quad . \quad (2)$$

y_1 is infinite when $\omega^2 = \frac{kg}{W}$; for lower values of ω the value of y_1 is positive, and hence the shaft is stable.

For values of ω greater than the critical value, y_1 becomes negative approaching the value $-h$., the shaft being again stable. This means that, as the speed is increased, the centre of gravity of the mass and the axis of rotation approach one another. The flexible shaft used in the De Laval turbine depends on this principle; the shaft being perfectly safe, as the whirling speed is well below the running speed, and fracture

does not take place in starting up, as the running speed passes through the critical value very rapidly.

If the ends of the shaft are fixed, it may be shown that

$$\omega = \sqrt{\frac{3(a+b)^3EIg}{Wa^3b^3}} \quad . \quad . \quad . \quad . \quad (3)$$

139. **Shaft with Uniformly Distributed Load.** Such a shaft when in the extreme position of deflection possesses zero kinetic energy and maximum strain energy, and when in passing through its static position the former is maximum and the latter zero. Lord Rayleigh has shown that, by equating the maximum strain energy to the maximum kinetic energy, the whirling speed may be calculated to a fair degree of accuracy.

Let w be the load per inch length (pounds).

ω the angular velocity of the shaft (rad./sec.) at the whirling speed.

Considering the shaft to be *freely* supported by its beahings, if y is the maximum deflection of an element of length δx at a distance x from one end,

Maximum strain energy of element

$$= \frac{w\delta x}{2} y$$

Total maximum strain energy of shaft

$$= \frac{w}{2}\int_0^l y dx \quad . \quad . \quad . \quad . \quad (4)$$

where l is the length of the shaft in inches.

The maximum velocity of the element occurs when passing through the static position, and is equal to ωy.

Hence, maximum kinetic energy of element

$$= \frac{w\delta x}{2g}(\omega y)^2$$

Total maximum kinetic energy of shaft

$$= \frac{w}{2g}\omega^2\int_0^l y^2 \, . \, dx \quad . \quad . \quad . \quad (5)$$

and equating the maximum kinetic energy to the maximum strain energy,

$$\frac{w}{2g}\,\omega^2\int_o^l y^2dx = \frac{w}{2}\int_o^l ydx$$

or

$$\omega^2 = \frac{g\displaystyle\int y\,.\,dx}{\displaystyle\int_o^l y^2\,dx} \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and $y = \dfrac{w}{24EI}\,(x^4 - 2lx^3 + l^3x)$ from (5), par. 86 (b).

Substituting this value of y and integrating

$$\omega^2 = \frac{24EIg}{wl^4}\,\frac{\frac{1}{5}}{\frac{31}{630}}$$

or

$$\omega = \frac{9{\cdot}87}{l^2}\sqrt{\frac{EIg}{w}} \quad . \quad . \quad . \quad . \qquad (7)$$

If Z is the maximum static deflection of the shaft

$$= \frac{5}{384}\,\frac{wl^4}{EI}$$

then from (7)

$$\omega = 9{\cdot}87\sqrt{\frac{5g}{384Z}}$$

or

$$\omega = \frac{22{\cdot}12}{\sqrt{Z}} \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

If the ends of the shaft are *fixed*, then

$$y = \frac{w}{24EI}\,(x^2l^2 - 2lx^3 + x^4)$$

and by substitution in (6) and integrating,

$$\omega = \frac{22{\cdot}45}{l^2}\sqrt{\frac{EIg}{w}} \quad . \quad . \quad . \quad . \qquad (9)$$

also $$Z = \frac{wl^4}{384EI}$$

$\therefore$ $$\omega = 22{\cdot}45\sqrt{\frac{g}{384Z}}$$

or $$\omega = \frac{22{\cdot}45}{\sqrt{Z}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

140. **Shaft with any number of Concentrated Loads.** If the shaft has free ends, and carries loads W_1, W_2, W_3, etc., at different distances from one end, then the empirical formula which follows may be used—

$$\frac{1}{\omega^2} = \frac{1}{\omega_s^2} + \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \text{etc.} \quad . \quad . \quad . \quad (11)$$

where ω = whirling speed of the system
ω_s = ,, ,, ,, shaft unloaded
ω_1 = ,, ,, ,, shaft carrying W_1 alone
ω_2 = ,, ,, ,, ,, ,, W_2 ,,
ω_3 = ,, ,, ,, ,, ,, W_3 ,,

141. **Unloaded Shaft.** The unloaded shaft may be treated as a shaft carrying a uniformly distributed load, or may be treated as follows—

Let w be the weight in pounds of unit length of the shaft.

Centrifugal force per unit length of shaft $= \frac{w}{g}\omega^2 y$ where y is the deflection from the axis of the shaft (inches) at a distance x in. from one end.

From par. 86 (1), $\frac{d^2y}{dx^2} = \frac{M}{EI}$, from which it follows that $\frac{d^4y}{dx^4} = \frac{w}{EI}$ where w is the load per unit run.

In the case of the unloaded shaft, due to rotation, the load per unit length is $\frac{w}{g}\omega^2 y$.

$\therefore$ $$\frac{d^4y}{dx^4} = \frac{1}{EI}\frac{w}{g}\omega^2 y$$

or $$\frac{d^4y}{dx^4} - b^4 y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $$b^4 = \frac{w\,\omega^2}{g} \cdot \frac{1}{EI} \qquad . \quad . \quad . \quad . \quad (13)$$

The solution of (12) is given by

$$y = A \sin bx + B \sinh bx + C \cos bx + D \cosh bx \qquad . \quad . \quad . \quad . \quad (14)$$

Assuming free ends, at each end $y = 0$ and $\frac{d^2y}{dx^2} = 0$.

Hence, $C = D = 0$.

$$\therefore \qquad y = A \sin bx + B \sinh bx \qquad . \quad . \quad . \quad . \quad (15)$$

at $x = l$ for $\frac{d^2y}{dx^2} = 0$, $A \sin bl = 0$, $\therefore A = 0$ or $\sin bl = 0$.

If $A = 0$ there is no deflection, and taking $\sin bl = 0$, we have $bl = \pi$, 2π, etc.

The lowest or fundamental value is

$$bl = \pi$$

$$\therefore \qquad b^4 = \left(\frac{\pi}{l}\right)^4$$

and from (13)

$$\frac{w}{g}\,\omega^2\,\frac{1}{EI} = \left(\frac{\pi}{l}\right)^4$$

and $$\omega = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{gEI}{w}} \qquad . \quad . \quad . \quad . \quad (16)$$

A result similar to that obtained in par. 139 (7).

By the method of equating the maximum strain energy to the maximum kinetic energy, it may be shown that the frequency of *transverse vibration* for each of the above cases is equal to the value of ω.

EXAMPLE 3.

A shaft of 2 in. diameter is freely supported in bearings 100 in. apart. A pulley weighing 50 lb. is mounted on the shaft mid-way between the bearing. What is the critical speed of the system? What would be the critical speed if a 30 lb. pulley be placed at a distance of 10 in. from each bearing? $E = 30 \times 10^6$ lb./sq. in.

Whirling speed of unloaded shaft $= N_s$

$$= \frac{60}{2\pi} \frac{9{\cdot}87}{l^2} \sqrt{\frac{gEI}{w}}$$

$$= \frac{60}{2\pi} \frac{9{\cdot}87}{10{,}000} \sqrt{\frac{12 \times 32 \times 30 \times 10^6 \times \pi}{4 \times \pi \times 0{\cdot}28}}$$

$$= 953 \text{ r.p.m.}$$

Whirling speed for 50 lb. pulley (assuming shaft weightless) $= N.$

$$= \frac{60}{2\pi} \sqrt{\frac{3(a + b)EIg}{Wa^2b^2}}$$

$$= \frac{60}{2\pi} \sqrt{\frac{3 \times 100 \times 30 \times 10^6\pi \times 12 \times 32}{4 \times 50 \times 2500 \times 2500}}$$

$$= 888.$$

Let N = whirling speed of the combined system.

$$\frac{1}{N^2} = \frac{1}{N_s^2} + \frac{1}{N_1^2}$$

$$= \frac{1}{(953)^2} + \frac{1}{(888)^2}$$

$$N = 650 \text{ r.p.m.}$$

Whirling speed for one 30 lb. pulley (assuming shaft weightless) $= N_2$.

$$= \frac{60}{2\pi} \sqrt{\frac{3 \times 100 \times 30 \times 10^6\pi \times 12 \times 32}{4 \times 30 \times 100 \times 8100}}$$

$$= 3185 \text{ r.p.m.}$$

Let N^1 = whirling speed of new system.

$$\frac{1}{(N^1)^2} = \frac{1}{N_s^2} + \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_2^2}$$

$$= \frac{1}{(953)^2} + \frac{1}{(888)^2} + \frac{2}{(3185)^2}$$

$$N^1 = 623 \text{ r.p.m.}$$

EXAMPLES XIV

1. Show that the hoop stress f in a revolving ring is given approximately by the expression $f = \frac{\rho v^2}{g}$ where ρ is the density of the material, and v is the velocity of the ring. Assuming this expression may be used to find approximately the mean stress in a thin cylinder turning about its axis, find the safe number of revolutions for a rotor 6 ft. in diameter if the stress is not to exceed 18,000 lb./sq. in. Take $\rho = 480$ lb./cub. ft. (Lond. Univ., 1913.)

Ans., 1,323 r.p.m.

2. Find the ring tension and the bending stress, in the rim of a flywheel, the section of which is rectangular, 12 in. wide and 7 in. thick. Diameter of wheel 8 ft., r.p.m. 200, six arms. Assume that owing to stretch of the arms, the bending stress in the rim segments is two-thirds of the value obtained, on the assumption that the rim segments are straight built in beams. (Lond. Univ., 1914.) *Ans.*, 684 lb./sq. in., 1,710 lb./sq. in.

3. A disc of 12 in. diameter has a central hole of 2 in. diameter and runs at 4,000 r.p.m. Calculate the hoop stress at the inner and outer radius given that $E = 30 \times 10^6$ lb./sq. in. and $w = 0{\cdot}28$ lb./cub. in.

Ans., 3,800 lb./sq. in., 970 lb./sq. in.

4. The rotor of a De Laval steam turbine is 24 in. diameter at the blade ring and is 3·75 in. thick at the centre. Calculate the thickness at the blade ring if the r.p.m. are 10,000, the stress uniform and equal to 25,000 lb./sq. in. and the density of the material 0·28 lb./cub. in. *Ans.*, 0·375 in.

5. The critical speed of a shaft supported in end bearings and loaded in any way is in many practical calculations taken to be expressed by $N = \frac{200}{\sqrt{d}}$ where N is the critical speed in revolutions per minute and d is the maximum static deflection in inches. With this assumption show that the maximum bearing span of L ft. for a hollow steel shaft of external diameter D_1 in. and internal diameter D_2 in., running at M r.p.m. is $L = K\left(\frac{D_1^2 + D_2^2}{M^2}\right)^{0{\cdot}25}$ Estimate the value of K if the maximum safe running speed is 30 per cent below the critical value. Weight of steel 0·28 lb. per cub. in. $E = 30 \times 10^6$ lb./sq. in. (Lond. Univ., 1921.)

6. The rotor of a De Laval steam turbine weighs 4 lb. and the shaft has a diameter of $\frac{3}{8}$ in. The bearings are $11\frac{1}{2}$ in. span and the wheel is 6 in. from the centre of one bearing. Calculate the critical speed of the system.

Ans., 2,773 r.p.m.

CHAPTER XV

TESTING AND TESTING MACHINES

142. THE machines used to test materials to destruction in tension, compression, and shear may be divided into two classes: one class in which the material is stressed by the application of a hydraulic load, whilst in the other class stressing is caused by power-operated screw gearing. In each case the load carried by the specimen under test is measured by a movable weight and lever, or system of levers. An example of each machine will now be described.

143. **Vertical Single-lever Testing Machine.** Fig. 103 shows a machine made by Messrs. J. Buckton & Co., Ltd., Leeds, capable of exerting a force of 50 tons. It consists of a vertical column, or frame, and a horizontal lever or steelyard *C*, resting on knife edges at *O*. The specimen *A* is gripped by the upper shackle *B*, which is suspended from the knife edge *P* of the steelyard *C*. The steelyard measures the forces applied to the specimen by the movement of the poise weight *D*; this movement being caused by the rotation of handwheel *Q*, and consequent rotation of screw *R* by means of the gear wheels *S*. The scale *E* and vernier *F* allow of very accurate readings of the applied forces.

The pulling force is applied by hydraulic pressure acting on a piston (Fig. 104), the piston being attached to the shackle *G*, which, in turn, grips the lower end of the specimen. Before commencing a test, the steelyard should be balanced; this condition being obtained by moving the poise weight until the steelyard floats equidistant between the stops *JJ*, this being indicated by the pointer *K*. The vernier *F* should now correspond with the zero of scale *E*.

In order to carry out a tensile test, one end of the specimen is gripped in the upper shackle *B* by means of wedge grips *LL*, Fig. 103A. The poise weight is advanced until a reading of about one-tenth of a ton is indicated. This puts on a small load, and facilitates the gripping of the specimen. The grips are now put into shackle *G* and held in position while the shackle is gently lowered by means of the hydraulic gear. The grips bite into the specimen, and the beam is pulled into

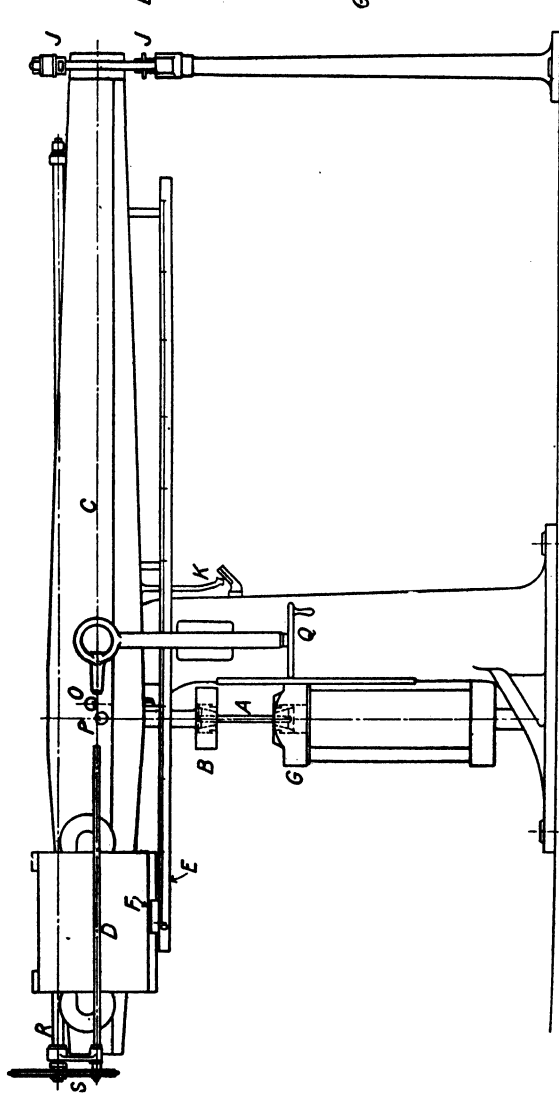



FIG. 103

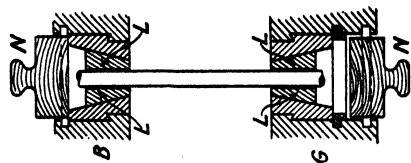



FIG. 103A

a horizontal position when the pulling gear is stopped. The tackle at N is now fixed in position in order to prevent the grips falling out when the specimen breaks. The test is begun,

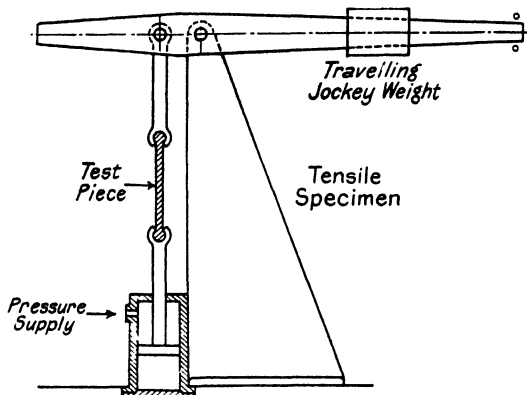



Fig. 104

the poise weight being advanced at a suitable rate, and the stretch of the specimen being taken up by the straining gear in such a manner that the steelyard is kept in a more or less

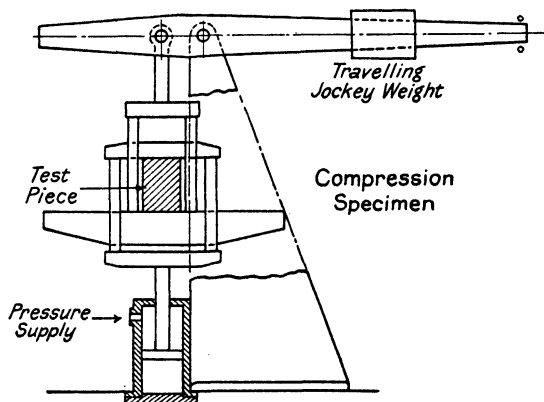



Fig. 105

horizontal position. This procedure continues until the specimen is broken.

In Figs. 104, 105, and 106 the machine is shown diagrammatically arranged for tensile, compression, and deflection testing respectively.

For tensile testing the grips used are taper, as shown at *LL*, but in the case of a cylindrical specimen a V-notch is cut in

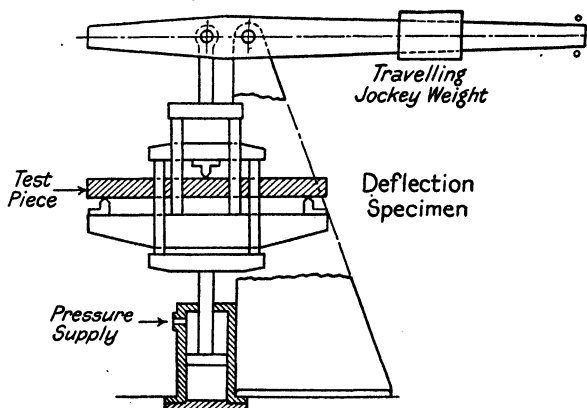



FIG. 106

the face of each grip to accommodate the specimen, and each grip has serrations on the gripping face to facilitate a secure hold of the specimen. For brittle materials, such as cast iron, the specimen is screwed at the ends, hence a screwed chuck is used. For brittle materials, a special self-centring arrangement must be employed, since bending will occur if the line of action of the load does not coincide with the geometrical axis of the specimen. Brittle materials are weak in tension, and hence, if the load causes bending as well as direct stress, the stress distribution over the section will not be uniform, and a smaller load will cause the material to fail than that if the stress distribution were uniform, as obtained with a central load. The grips shown in Fig. 107 are common for the different types of testing machines, being designed to be self-centring. A spherical

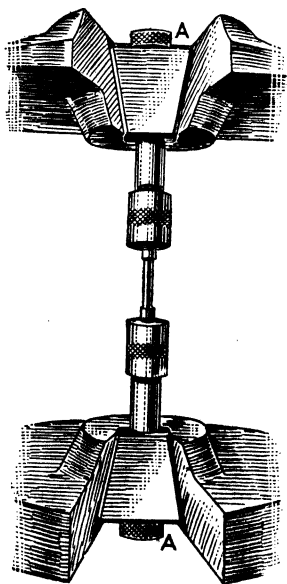



FIG. 107

seat is turned in the solid taper chuck at AA, and the head of the spindle carrying the screwed chuck is made to fit the spherical seat; thus, the line of pull will accommodate itself to the axis of the specimen.

144. **Vertical Screw Power Testing Machine.** Fig. 108 shows a diagrammatic view of a machine of this type, as manufactured

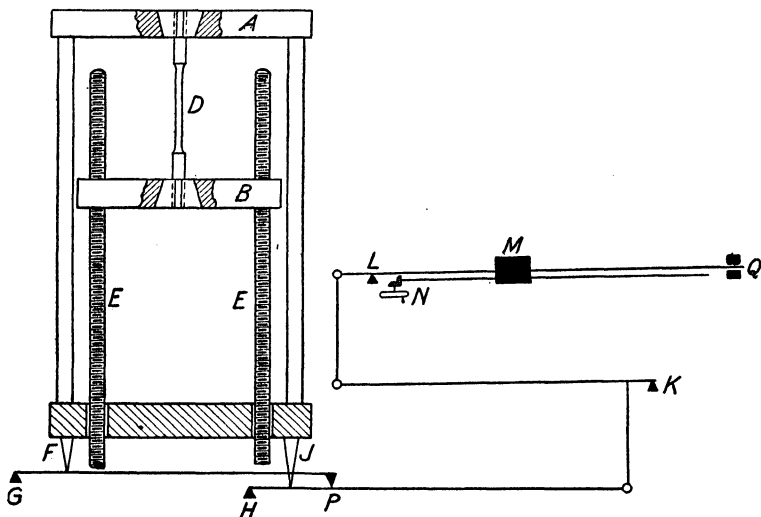



FIG. 108

by Messrs. Riehlé Bros., Philadelphia. It consists of a rectangular framework, to the upper end A of which the specimen is fastened; the other end of the specimen being caught in jaws, held by the movable head B, which is threaded on powerful screws EE, these screws passing loosely through the lower portion of the framework C. The framework has projecting knife-edges, F and J, which rest on levers pivoted at G and H respectively. A further system of levers pivoted at K and L enables the pull on the specimen to be balanced by a small jockey weight M, mounted on a graduated arm, the jockey weight being operated by a screw and gearing N. The screws EE are operated by motor-driven gearing. The pull exerted by the screws is transmitted through the specimen to the framework, and is measured by moving the jockey weight along the graduated arm, the arm being kept floating between the stops Q. These machines are made with two, three, or

four screws, and a machine of the three-screw type is shown by Fig. 109, which is capable of exerting a pull or push of 150,000 lb. The three-screw arrangement ensures equality of distribution of load, and the screw nearest the graduated arm runs in a direction opposite to that of the other screws. This does away with any tendency to rotate the pulling head.

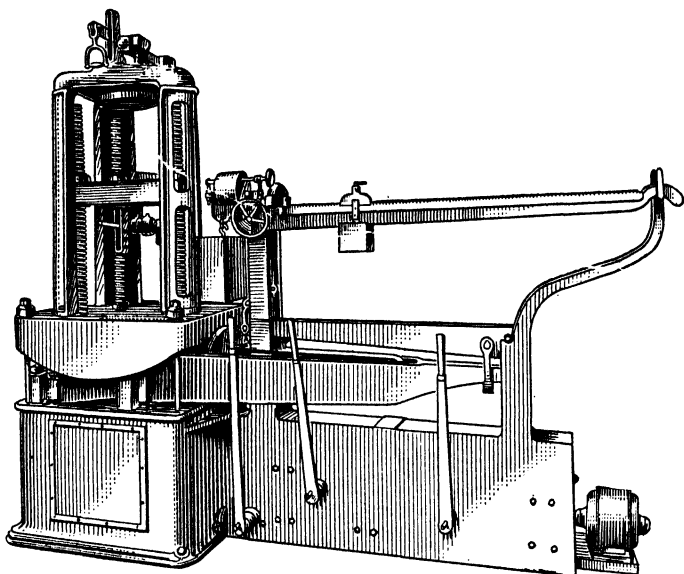



Fig. 109

The machine can be furnished with six speeds, and a suitable clutch allows of reverse motion.

145. **Extensometers.** In order to measure the minute extensions of a test bar for stresses up to the elastic limit, instruments called extensometers are used. These instruments magnify the stretch of the specimen, either by optical or mechanical means, and the stretch can be measured with great accuracy. An instrument of each type will now be described.

146. **Cambridge Extensometer.** This instrument is of the mechanical magnification type, and is shown diagrammatically by Fig. 110, and more fully by Figs. 111 and 112. It is capable of measuring accurately up to $\frac{1}{1000}$ of a millimetre, but readings can be taken with a fair degree of accuracy up to $\frac{1}{10000}$

of a millimetre. The instrument possesses the following advantages—

(1) The readings can be taken with ease and rapidity.

(2) It is strongly made; is very simple, and has no parts likely to get out of order.

(3) It consists of two free parts, so that if a specimen unexpectedly breaks, the extensometer is not seriously damaged.

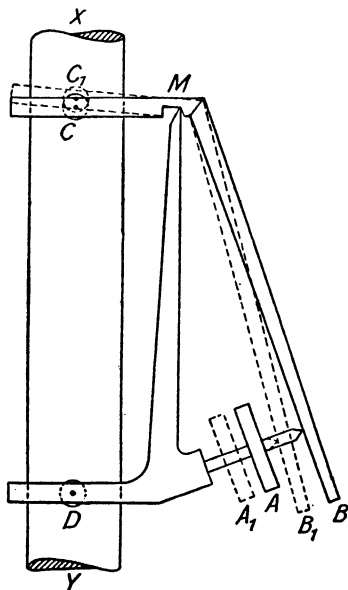



Fig. 110

This instrument has been designed to resemble a workshop tool rather than a scientific instrument. Having no delicate parts, mirrors, or microscopes, it is not easily damaged, and yet it gives very accurate readings.

The instrument is made in two separate pieces each of which is separately attached to the test-piece *M* (Figs. 111 and 112) by hard steel conical points *P*, *P* and *P′*, *P′*. The steel rods carrying these points slide in geometric slides and after being driven gently into the centre punch marks in the test-piece are clamped in position by the milled heads *R*, *R*. Both parts of the instrument should be capable of rotating quite freely about the points, but there must be no backlash.

The lower piece carries a micrometer screw fitted with a hardened steel point X and a divided head H. It also carries a vertical arm B at the top of which is a hardened steel knife-edge. The upper and lower pieces work together about this knife-edge. A nickel-plated flexible steel tongue A forming a continuation of the upper piece is carried over the micrometer point X. This tongue acts as a lever magnifying the extension

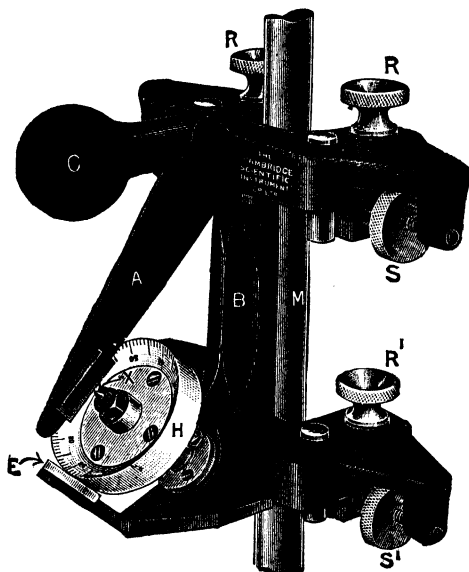



FIG. 111

of the specimen so that the movement of the steel tongue to or away from the steel point X is five times the actual extension of the specimen.

To take a reading with the extensometer the thin steel tongue A is caused to vibrate and the divided head then turned till the point X just touches the hard steel knife-edge on the tongue as it vibrates to and fro. This has proved to be a most delicate method of setting the micrometer screw, as the noise produced and the fact that the vibrations are quickly damped out indicate to $\frac{1}{1000}$ mm. the instant when the screw is touching the tongue. After the load is applied a second

reading is taken in a similar manner and the difference in the readings gives directly the extension of the testpiece.

The standard instrument is suitable for use on specimens up to 20 mm. or $\frac{3}{4}$ in. diameter, the centre points P, P' being 100 mm. apart.

The pitch of the micrometer screw is $\frac{1}{2}$ mm. and the head is divided into 100 parts. As the lever multiplies five times,

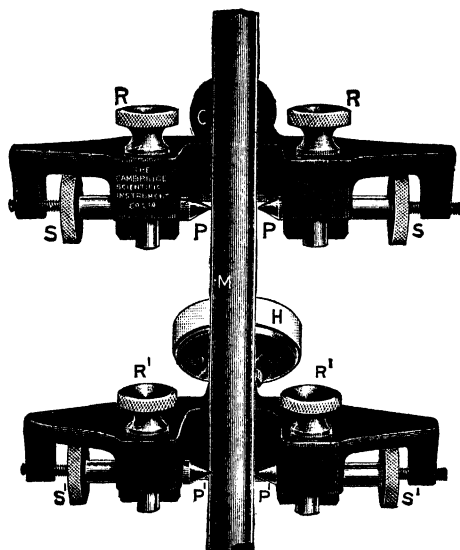



Fig. 112

each division on the head corresponds to an extension of the test-piece of $\frac{1}{1000}$ mm., and as the tenths of divisions can be estimated by eye, readings can be taken to $\frac{1}{10,000}$ mm., although it is not claimed that the results are trustworthy to this degree of accuracy. The effective length of the test-piece being 100 mm., readings can be taken to $\frac{1}{1,000,000}$ of the length of the test-piece by estimation.

A marking-off tool is supplied (Fig. 113) for easily and very accurately marking off the test-pieces for the extensometer. It consists of a cast-iron base having two V grooves running

lengthwise along it, their centre lines being exactly the same distance apart as the centre points P, P'. These grooves are cut away mid-way along their length, permitting the test-piece

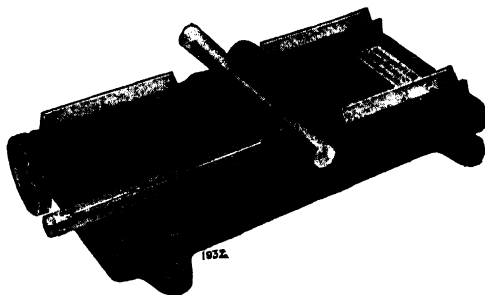



FIG. 113

to be clamped in another groove which runs across and at right angles to the first groove. A hardened steel centre punch

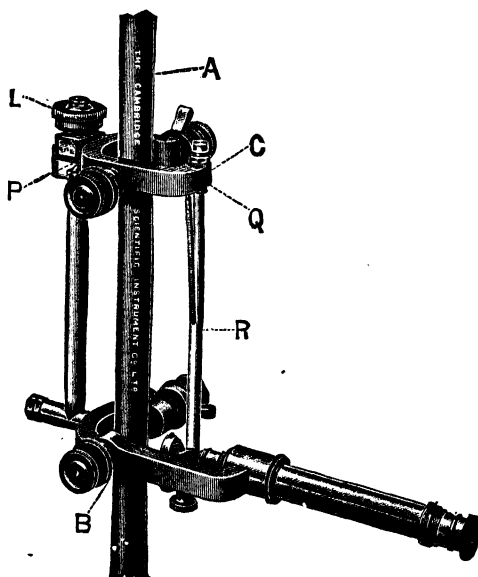



FIG. 114

slides in the grooves, and by tapping it with a hammer the test-piece is truly marked off. The point of the centre punch has three small flats on it so that it actually makes a triangular

hole in which the conical points of the extensometer fit geometrically without any play.

147. **Ewing Extensometer.** The Ewing Extensometer, designed by Sir J. A. Ewing, F.R.S., is shown by Figs. 114 and 115, and is an example of an optical magnification instrument. This extensometer allows the stretching of the specimen to be continuously watched. It is applicable to large or small

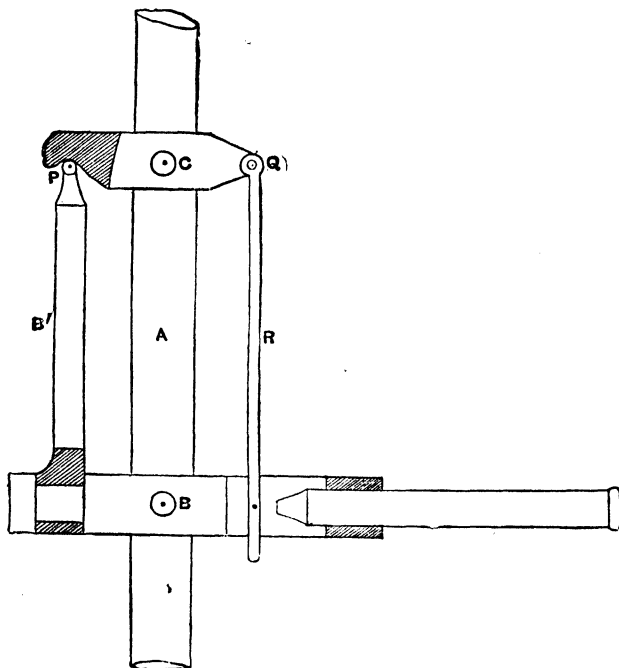



FIG. 115

test-pieces, and can be used on testing machines of either the vertical or the horizontal type.

The extensometer is clamped to the specimen by two pairs of set screws with steel points, which grip the bar, the distance apart of the two points being accurately adjusted so that a definite length of the specimen is under observation. The upright rod projecting from the lower part of the extensometer ends in a rounded point, which engages with a conical hole in the upper half of the instrument. When the test-piece extends,

this point serves as a fulcrum for the upper part of the extensometer, so that a point on the opposite side of the test-piece, and the same distance from it as the fulcrum, will move, relatively to the lower half of the instrument, through a distance equal to twice the extension. This distance is measured by means of a microscope attached to the lower clip, the movement being brought into the field of view of the microscope by means of a rod hanging from the upper clip and carrying at its lower end a fine wire on which the microscope is sighted. The displacement is read on a micrometer scale in the eyepiece. The standard pattern extensometer is designed for use with the two clamps in centres 200 mm. apart, and extra parts can be supplied enabling it to be converted into an instrument suitable for use with centres 50 mm. apart. Other patterns are supplied to read in English units with centres either 8 in. or 2 in. apart. By estimation, readings of the extension can be taken to $\frac{1}{2000}$ mm. or $\frac{1}{50000}$ in. To ensure correct readings the microscope must be adjusted to give constant magnification. The screw with divided head seen on the upper part of the extensometer enables this adjustment to be tested, since when the microscope is in correct adjustment a single turn of the divided head should cause a displacement of the wire through 50 divisions on the eyepiece scale.

The adjusting screw further serves to bring the sighted mark to a convenient point on the eyepiece scale, and also to bring the mark back if the strain is so large as to carry it out of the field of view. In this case a single turn of the screw adds another 50 divisions to the range shown on the eyepiece scale.

To facilitate the application of the extensometer to any rod, a clamping bar is added by which the two clips are held at the correct distance apart, with the axes of their set screws parallel, while they are being secured to the test-piece.

A modified form of Ewing extensometer can also be supplied for measuring the elastic compression of short blocks, having clip centres 50 mm. apart, or 2 in. apart in the pattern designed for English readings. Readings can be taken to correspond to $\frac{1}{5000}$ mm. or $\frac{1}{125000}$ in. of compression in the two patterns. Both of the compression forms are suitable for tension tests on lengths of 50 mm. and 2 in. respectively.

A recent extensometer by Prof. Lamb, which can be adopted for tensile or compression tests, will be found in the paragraph dealing with concrete.

EXAMPLE 1.

The following observations were obtained during a tensile test of a piece of steel. The readings of the extension were taken by means of a mirror extensometer on a gauge length of 6 in. The extensometer constant was 250,000 (i.e. a reading of 1 by the extensometer was equivalent to an extension of $\frac{1}{250000}$ in.)—

Load (Tons)	Extension on 6 in. in Extensometer Readings	Other Data obtained from Test	
0·0	0		
0·5	220	Diameter of test piece	= 0·562 in.
1·0	440		
1·5	670	Cross-sectional area of test piece	= 0·2481 sq. in.
2·0	900		
2·5	1130		
3·0	1350		
3·5	1570	Diameter at fracture	= 0·379 in.
4·0	1800		
4·5	2025	Yield load	= 9·6 tons
5·0	2250		
5·5	2470	Maximum load	= 13·24 tons
6·0	2705		
6·5	2945	Extension (including fracture) on 2 in.	= 0·49 in.
7·0	3190		
7·5	3470		
8·0	3780		

Plot the curve, and estimate from the plotted observations the limit of proportionality and modulus of elasticity in tons per square inch. Give any other results which can be obtained from the data given above, and which are usually obtained from a commercial tensile test. (A.M.I.C.E., 1926.)

In Fig. 116 the load is shown plotted against extensometer readings, and it will be observed that the curve bends over at a load of 7 tons.

$\therefore$ Limit of proportionality $= \dfrac{7}{0{\cdot}2481} = 28{\cdot}2$ tons/sq. in.

The extension represented by $BC = \dfrac{1950}{250{,}000}$ in.

$$\text{Strain} = \frac{1950}{250{,}000 \times 6}$$

$$\text{Stress} = \frac{AB}{0{\cdot}2481} = \frac{4{\cdot}3}{0{\cdot}2481} \text{ tons/sq. in.}$$

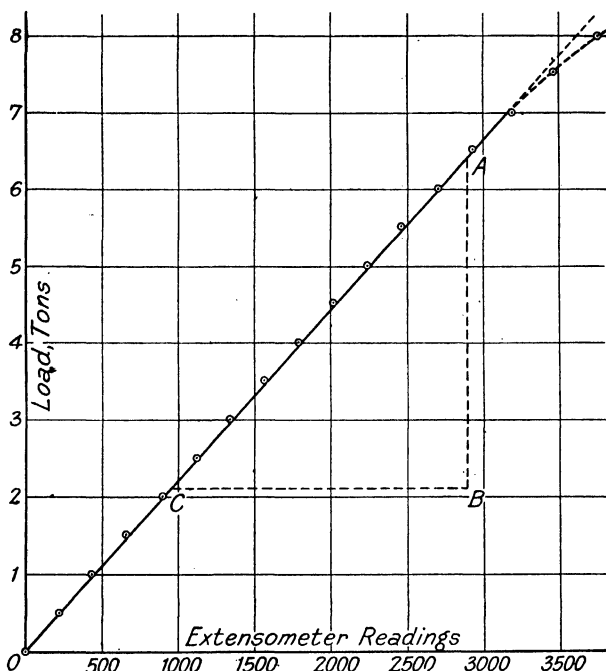



FIG. 116

$$\therefore \text{Modulus of elasticity} = \frac{\text{stress}}{\text{strain}}$$

$$= \frac{4{\cdot}3}{0{\cdot}2481} \times \frac{250{,}000 \times 6}{1950}$$

$$= 13\ 330 \text{ tons/sq. in.}$$

Area of cross-section at fracture $= 0{\cdot}7854 \times (0{\cdot}379)^2$

$= 0{\cdot}1129$ sq. in.

Percentage reduction in area $= \dfrac{0{\cdot}2481 - 0{\cdot}1129}{0{\cdot}2481} \times 100$

$= \dfrac{0{\cdot}1352}{0{\cdot}2481} \times 100$

$= 54{\cdot}4$

Percentage elongation on 2 in. gauge length $= \dfrac{0{\cdot}49}{2} \times 100$

$= 24{\cdot}5$

Yield stress $= \dfrac{9{\cdot}6}{0{\cdot}2481}$

$= 38{\cdot}68$ tons/sq. in.

Maximum stress $= \dfrac{13{\cdot}24}{0{\cdot}2481}$

$= 53{\cdot}3$ tons/sq. in.

148. **Extensometers for Wires.** The apparatus shown in Fig. 117 has been designed by Searle to measure the extension of a wire subjected to a vertical tensile load. It consists of two frames, *A* and *B*, connected by a link *C* working between centres. The frame *A* is suspended by a wire *D*, securely fastened at its upper end, and held at each end by a chuck. A constant load *F* keeps the wire *D* taut. The wire *E*, under test, is similarly held and connected to the frame *B*. Loads are applied to *E* by means of a scale pan *G*. A spirit level *H* is attached to *A* by means of pivots, and the other end rests on the point of a micrometer screw in frame *B*. Before loading the wire, the bubble of the level is brought to zero, and the reading on the scale pan and micrometer noted. The load is applied and the bubble again brought to zero. The difference between the new reading of the micrometer and the former reading is the extension of the wire. The screw has a pitch of 1 mm., which is very accurately cut, and the head is divided into 100 parts.

149. Variation of Strength and Ductility. The chemical composition of a material, and the heat treatment to which

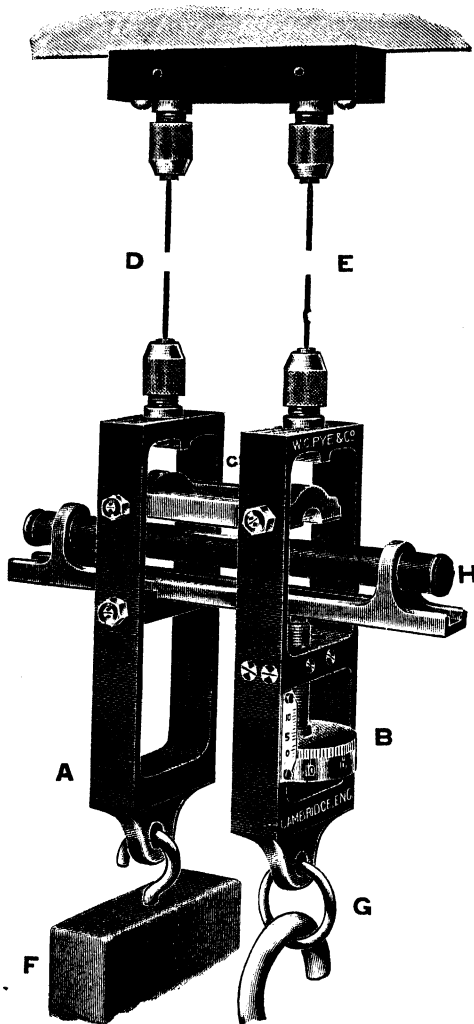



Fig. 117

it has been subjected, have a great effect on the strength of the material. The strength is also affected by the shape of

Analysis	Carbon . . .	0·130	0·190	0·270	0·290	0·340	0·380	0·450	0·510	0·570	0·630	Per cent
	Silicon . . .	0·028	0·037	0·065	0·037	0·055	0·066	0·167	0·083	0·121	0·111	”
	Sulphur . .	0·048	0·034	0·038	0·046	0·023	0·030	0·040	0·038	0·034	0·038	”
	Phosphorus . .	0·056	0·058	0·055	0·057	0·045	0·025	0·058	0·053	0·058	0·055	”
	Manganese . .	0·400	0·620	0·580	0·750	0·680	0·690	0·610	0·590	0·600	0·560	”
Tensile Test	Proportionality limit	12·650	13·100	15·000	16·000	16·450	13·790	17·500	16·600	20·450	22·500	Tons/sq. in.
	Yield stress . .	17·220	17·190	18·500	20·400	20·900	21·800	24·000	23·800	24·350	26·250	”
	Maximum stress .	24·700	29·000	32·700	34·800	37·200	39·400	42·800	44·700	46·700	51·400	”
	Breaking stress .	17·950	21·300	26·200	28·000	30·200	31·000	39·400	44·000	44·600	51·400	”
	Elongation . .	30·000	29·900	24·900	25·000	23·750	22·500	18·800	16·850	15·200	12·250	Per cent
	Contraction . .	69·000	60·800	55·600	57·500	56·200	54·400	36·000	26·200	30·600	14·620	”
	Hardness number .	116	127	145	157	162	171	191	197	229	233	Kg./sq. cm.

the specimen under test, and by other factors which will now be considered.

150. **Effect of Carbon on Steel.** The mechanical properties of steel are very largely influenced by the amount of carbon in the steel, and the table* on page 303, consisting of tests on a complete family of steels, shows the variations very clearly.

It will be noted that the strength of the steel increases with increase in the carbon content, but the ductility decreases as the carbon content is increased. In Fig. 118 stress is shown plotted against percentage elongation, and in Fig. 119 all the results in the table are shown plotted to a base of carbon content. It is worthy of notice that there are apparently simple linear relations between a number of the quantities.

151. **Effect of Hardening.** A steel may be hardened by heating it to a high temperature, and then rapidly cooling it in a cold liquid. The strength of the steel is greatly increased by this process, but its ductility is greatly reduced, and it becomes very brittle. Usually further heat treatment is required before the steel is of commercial use. The amount of hardness is influenced by the rate at which quenching takes place, and also by the carbon content. The following table contains a list of case-hardening steels, and shows the variations of strength and ductility with hardening. Each steel was tested before hardening, that is in the rolled condition. It was then hardened by heating to about 800° C., soaked for three-quarters of an hour and quenched. In par. 153 the result of further heat treatment on such a steel is shown.

Maker's Identification Mark	Yield Point tons/sq. in.		Max. Stress tons/sq. in.		Percentage Elongation on 2 in. Gauge Length		Percentage Reduction in Area	
	Before	After	Before	After	Before	After	Before	After
K.E. 2301 . .	21·3	23·3	29·2	36·8	31·5	31·5	55·3	62·6
K.E. 287 . .	25·8	42·0	36·2	53·9	27·5	22·5	52·2	53·4
K.E. 128 . . .	26·9	54·5	39·3	66·0	28·5	17·0	61·2	51·0
K.E. 660 . .	36·0	68·0	48·0	82·0	23·0	14·0	50·0	46·5
K.E. 24D . .	31·5	68·0	44·2	78·0	26·0	14·0	52·0	40·0
K.E. 169 . .	38·5	77·0	55·0	90·0	14·0	14·5	31·1	52·0

* "Some Experiments on Fatigue of Metals," J. H. Smith, *Journal of Iron and Steel Inst.*, No. II, for 1910.

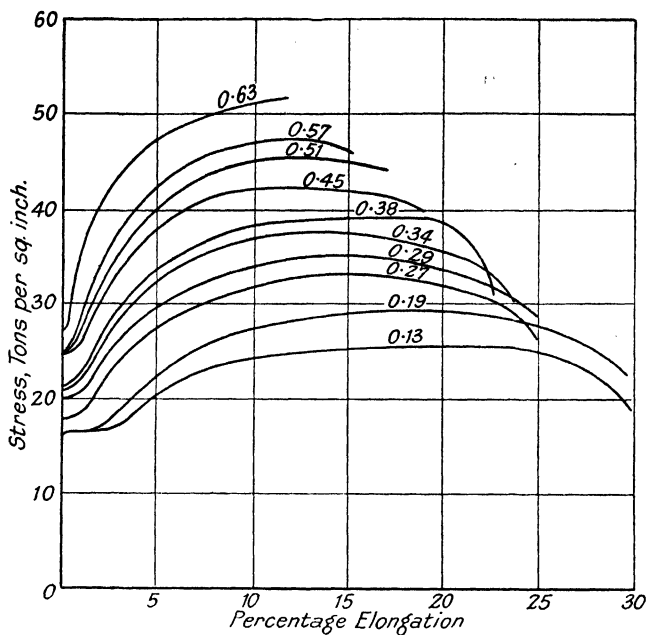



FIG. 118

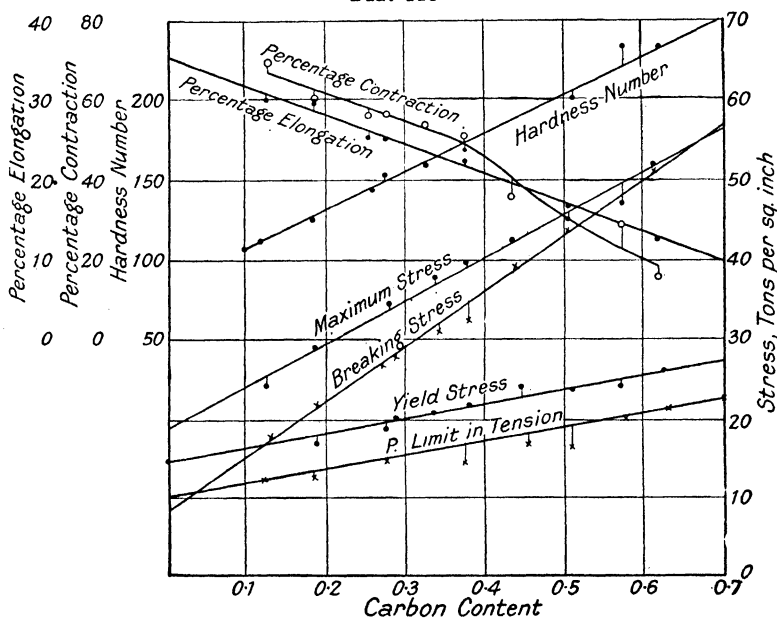



FIG. 119

152. **Effect of Annealing.** A steel which has been hardened, due to rapid cooling or to repeated straining processes, may be brought to its original state by heating to a blood-red heat and

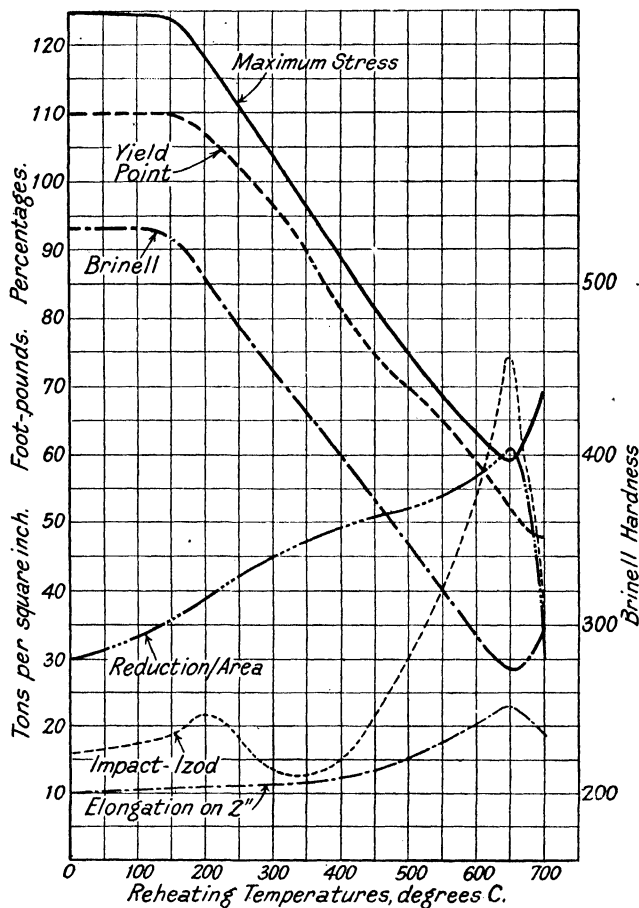



Fig. 120

cooling very slowly. Aluminium may be annealed by the same process, but copper and brass require to be cooled quickly, as slow cooling hardens them. The strength of a material is reduced by annealing, but the ductility is greatly increased.

153. **Effect of Tempering.** A material which has been hardened may be brought to a required degree of strength and

ductility by further heat treatment, or tempering. The following table shows the effect of tempering on a nickel-chrome steel—a steel which is of great use in automobile and aircraft work, and large connecting rods, etc. The steel was heated to 820° C., soaked for three-quarters of an hour and quenched in oil. It was then tempered for one hour at the temperatures stated, and cooled in oil.

Tempering Heat	Yield Point, tons/sq. in.	Maximum Stress, tons/sq. in.	Elongation % on 2 in.	Reduction of Area %	Brinell Hardness Number	Izod Impact, ft./lbs.
Hardened only .	110·0	125·0	10·0	30·0	532	16·0
150° C. . .	110·0	124·0	11·0	36·0	532	18·5
200° C. . .	106·5	119·0	11·0	39·0	512	22·0
250° C. . .	102·0	112·0	11·0	42·0	477	18·5
300° C. . .	96·5	104·0	11·5	45·0	460–444	14 0
350° C. . .	90·0	97·0	12·0	47·5	430	13·0
400° C. . .	81·5	89·0	12·5	49·5	402	15·0
450° C. . .	75·0	82·0	13·5	51·0	375	21·0
500° C. . .	70·0	75·0	15·5	52·5	351	30·0
550° C. . .	65·0	69·5	18·0	54·0	321	40·0
600° C. . .	59·5	64·0	20·5	56·5	293	53·0
650° C. . .	52·0	59·0	23·0	61·5	269	74·5
700° C. . .	48·0	70·0	18·5	30·0	302	36·5

The results are shown plotted to a base of tempering temperatures in Fig. 120, and it will be easily seen that increase in tempering temperature reduces the strength of the steel, but increases its ductility. It should also be noted that the strength of this steel is much above that of a mild steel. The addition of nickel, chromium, vanadium, and manganese adds greatly to the strength of a steel, but decreases ductility.

Referring to the table in par. 151, the steels are as follows—

K.E. 2301 A mild steel (free from nickel and chrome).
K.E. 287 }
K.E. 128 } A 3 per cent nickel chrome steel.
K.E. 660 A 5 per cent nickel chrome steel.
K.E. 24D A special nickel chrome steel.
K.E. 169 A nickel chrome vanadium steel.

The strength and ductility of other steels are given in par. 159.

154. **Effect of Low-temperature Heat Treatment.** If a ductile material be strained just beyond the elastic limit, and then immersed in water at 100° C. for about one hour, the elastic

properties are recovered, and, on restraining, the yield point is found to have a higher value.

155. **Effect of Period of Rest.** When a ductile material which has just been strained is given a very long period of rest, the elastic properties are regained; the yield point will have an increased value, and the ductility will be decreased.

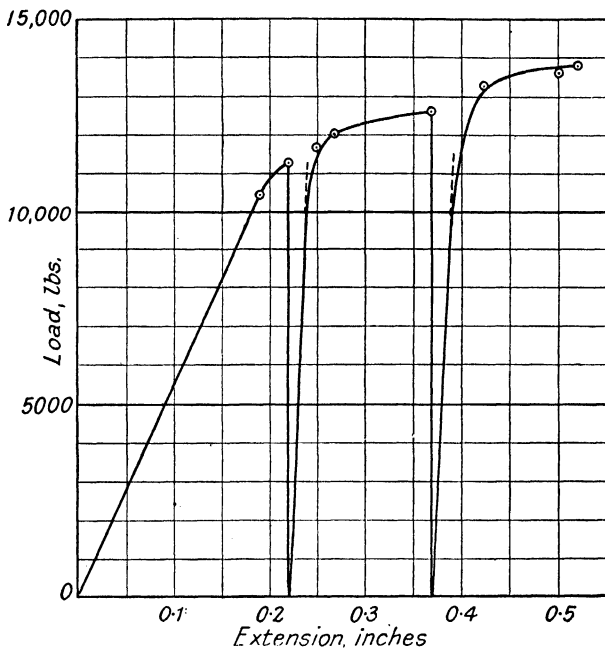



Fig. 121

The recovery of the elastic properties, by means of a period of rest, is a very much slower process than when water at 100° C. is used. The effect is more marked in the harder steels; immersion in water 100° C. for one hour being equivalent to a rest period of a number of years.

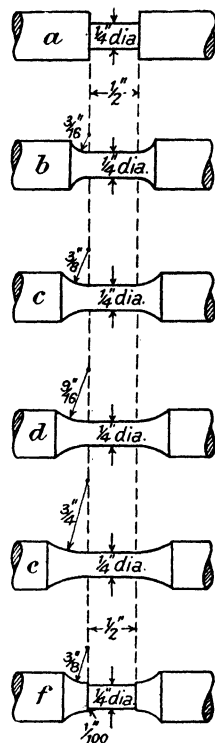
156. **Raising the Yield Point by Overstrain.** In testing ductile materials it is found that if they are strained up to the yield point, then unloaded and immediately restrained, the yield point is raised, but the elastic limit and range of proportionality are decreased (Fig. 121). If the loading and unloading process is continued, it is possible to make the yield point agree with the point of rupture.

157. **Mechanical Hysteresis.** When a specimen is loaded inside the elastic range, and then unloaded, the curves for each process do not agree, but form a loop. Part of the strain develops by creeping during the loading process after the load is applied, and portion of the strain disappears by creeping after the load has been removed. This phenomenon is known as "Mechanical Hysteresis."

158. **Effect of Shape of Specimen on Strength.** The enlarged ends of a specimen affect the yield stress and the ultimate stress if the specimen is short. When the material begins to become plastic, and flow, the material at the enlarged ends tends to go to the rescue of that portion of the specimen at which the waist is forming, thus a short specimen will give values for yield stress and maximum stress higher than those obtained from a long specimen. The ductility, as measured by elongation, and contraction will also be smaller in the former case. In order to obtain uniformity of results, it has been specified by the Engineering Standards Committee that the parallel portion of a specimen must be at least nine times the diameter of the specimen, and the elongation be measured on a length equal to eight diameters.

FIG. 122

The strength of the specimen is greatly affected by the rate of change of cross-section. When the change is abrupt, the stress is not evenly distributed over the cross-section where the change occurs, but is concentrated at the sharp angle. The strength will have a greatly reduced average value. This effect is more marked in the brittle materials. When a piece of material is subjected to a varying or alternating load, the effect of variation of cross-section is very marked indeed.

Fig. 122 shows the form of various specimens* which were subjected to a load which varied continuously from a given value in tension to an equal value in compression. The

* "Some Experiments on Fatigue of Metals," by J. H. Smith, *Journal of Iron and Steel Inst.*, 1910, No. II.

following table shows the value of the applied stress and the number of reversals which each specimen withstood until rupture occurred.

Specimen	Maximum Stress tons/sq. in.	Minimum Stress tons/sq. in.	Range of Stress tons/sq. in.	Reversals for Rupture
a	12·7	– 12·7	25·4	3,000
b	12·8	– 12·8	25·6	717,300
c	12·6	– 12·6	25·2	739,400
d	12·7	– 12·7	25·4	171,000
e	12·8	– 12·8	25·6	204,480
f	13·2	– 13·2	26·4	30,240

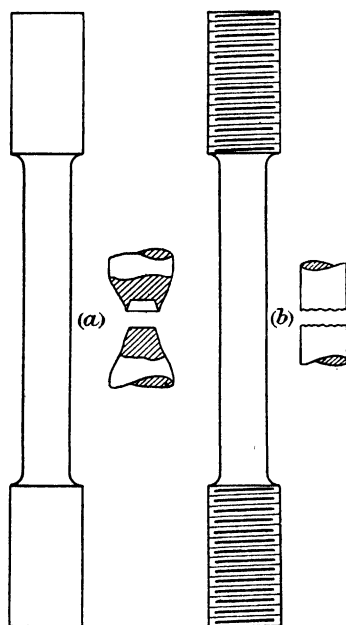



FIG. 123

It is the usual practice to turn down the shank of a bolt used for the "big end" of a connecting rod to a diameter equal to that at the bottom of the thread. This prevents localization of the stress at the cross-section where the thread ends.

The form of the specimen used for tensile testing is shown by Fig. 123, that at *a* being for a ductile material, and that at *b* for a brittle material. The brittle material requires to have a self-centring chuck, and to have screwed ends, as the taper jaws used for the ductile material would crush the ends of the brittle material and cause failure. The form of fracture for each is given at *a* and *b* (Fig. 123), the ductile material drawing down to a waist, and finally rupturing in what is called a " cone and cup " fracture. The brittle material shows very little extension, and breaks practically at right angles to the axis.

The reason for the " cone and cup " fracture is that the material really fails by shear. Experimental verification of this was obtained by Luder. He observed the behaviour of a highly-polished specimen as the load was applied, and at the yield point lines were observed on the surface, in a criss-cross formation at about 60° to the axis. These lines are called " Luder's Lines," and show that molecular slip is taking place in the direction of the shear stress, and by par. 15 the resistance to shear determines the strength of a material such as mild steel.

In order to test a wire rope, the ends of the test piece are opened out and bent in the form of a hook. Each end is placed in a conical mould, and lead poured in. The hooks take a secure grip in the lead and the ends can now be held in the testing machine.

159. **Effect of Temperature.** At low temperatures the tenacity of a mild steel is increased ; the material, however, becomes very brittle, its ductility being almost negligible.

With increase of temperature, the elastic limit decreases, and is accompanied by a corresponding decrease in the modulus of elasticity. The tenacity of the material decreases until a temperature of approximately 150° C. is reached. From this temperature the tenacity increases, reaching a maximum value in the neighbourhood of 300° C., after which it decreases as the temperature is further increased. The following table A* shows the effect of high temperature on the tenacity of various steels.

The ductility decreases with increase in temperature until a temperature of about 150° C. is reached. After this temperature the ductility increases with further increase of

* " Valve Steels," by P. B. Henshaw, *Journal of Roy. Aero. Soc.*, March, 1927.

TABLE A

Steel	Tensile Tests, Cold				Tensile Tests, Hot (°C.)							
	Elastic Limit, tons/sq. in.	Maximum Stress, tons/sq. in.	Elongation on 2 in., %	Reduction of Area, %	600	650	700	750	800	850	900	950
3% nickel chrome	48·5	56·5	23·0	62·5	34·0	23·6	13·5	11·2	9·2	6·9	4·5	—
Stainless steel	47·0	48·3	24·0	58·1	24·2	17·3	10·2	18·0	6·1	5·6	7·9	—
Silicon chrome	51·0	63·0	21·0	40·0	42·0	34·5	25·3	17·8	10·7	7·2	3·9	3·0
Chrome steel	42·0	54·0	24·5	55·0	36·5	30·0	19·0	13·0	7·0	7·2	7·3	5·2
Cobalt chrome	42·0	58·0	13·0	22·0	45·0	30·5	24·8	13·6	10·0	5·8	8·1	6·0
High speed steel	46·0	61·0	15·0	24·0	41·0	26·6	21·3	16·8	9·8	7·7	8·5	5·6
High nickel chrome	42·0	68·0	27·0	45·0	42·6	38·3	33·8	28·6	24·0	19·4	15·0	12·5

TABLE B. R.R. 53

Temperature	Ult. Stress, Tons/sq. in.		Brinell at Temperature		Brinell after Cooling	
	Sand Cast	Die Cast	Sand Cast	Die Cast	Sand Cast	Die Cast
20° C.	18·5	24·0	129	138	—	—
200° C.	16·0	22·0	101	115	129	138
250° C.	14·0	19·5	78	80	121	129
300° C.	13·0	15·5	50	55	101	105
350° C.	8·8	9·0	27	30	70	85

TABLE C. R.R. 59

Temperature	Ult. Stress Tons/sq. in.	Brinell at Temperature	Brinell after Cooling
20° C.	28·0	134	—
200° C.	21·5	110	134
250° C.	19·5	87	125
300° C.	13·0	52	90
350° C.	8·0	27	75
400° C.	—	12	70

temperature. The increase is not regular, however, but takes place in an erratic manner. Tables B and C show the effect of temperature on a cast and wrought aluminium alloy respectively.

160. Compression Testing. For compression testing, the machine is fitted with two hardened steel blocks, one of which rests in a spherical seating to ensure central loading. The specimen usually consists of a short cylinder, about $\frac{3}{4}$ in. diameter and $1\frac{1}{2}$ in. long. If this ratio of length to diameter be exceeded, there is a danger of failure being due to both compression and bending.

Ductile materials fail as shown at Fig. 124 (*a*), cracks opening parallel to the axis of the specimen. Since the area of the

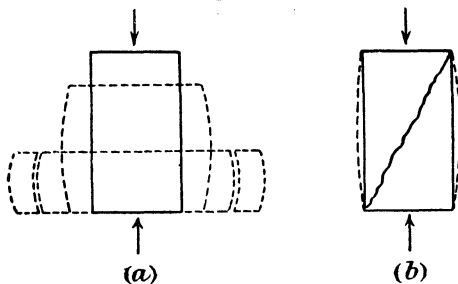



FIG. 124

cross-section is continually changing with increase of load, the area may be found as follows—

Let a_1 and l_1 be the mean area and length respectively at the commencement of the test, and let a_2 and l_2 be the corresponding values when a certain load is applied. Then, since the volume remains constant

$$a_2 = a_1 \frac{l_1}{l_2}$$

The fracture of a brittle specimen is shown by Fig. 124 (*b*). The failure is really due to shear, for the reason given in par. 15, the inclination of the fracture being about 35°–40° to the axis of the specimen. In the "cone and cup" fracture, and in the cast iron fracture, the theoretical value of the angle of inclination is 45° (par. 15), but the inclination is effected by the normal stress on the planes. In Fig. 9, taking μ as the coefficient of friction, the actual shear resistance

$$f_s = \frac{W}{A} \sin \theta \cos \theta = q + \mu \frac{W}{A} \cos^2 \theta$$

where q is the ultimate cohesive resistance to *pure* shear, i.e. when no normal force is exerted.

Fracture will occur on the section over which q is a maximum, i.e. when $\frac{dq}{d\theta}$ is maximum.

$$q = \frac{W}{A}(\sin\theta\cos\theta - \mu\cos^2\theta)$$

and $\frac{dq}{d\theta} = o$ gives $-\mu = \cot.2\theta$

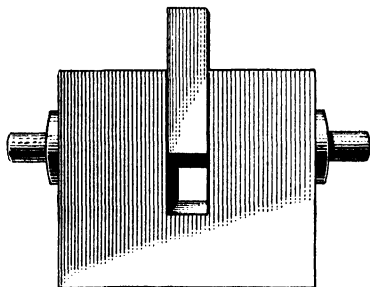



FIG. 125

If ϕ is the angle of friction, then

$$-\mu = -\tan.\phi = \cot.\left(\frac{\pi}{2} + \phi\right)$$

$$\therefore \qquad 2\theta = \frac{\pi}{2} + \phi$$

and $$\theta = \frac{\pi}{4} + \frac{\phi}{2}$$

For cast iron ϕ is approximately $20°$

$$\therefore \qquad \theta = 55°$$

The above theory is due to Navier.

161. **Shear Testing.** The shearing tool shown in Fig. 125 is arranged for making both single and double shear tests. The block which carries the specimen rests on the table of the testing machine, and the movable head (in the case of the Riehle machine) carrying a crushing tool, forces the knife through the specimen. The block for holding the specimen has a slot for guiding the knife. A large hole in the block is fitted with two hard steel bushings held in place by knurled nuts.

For double shear the specimen is slipped through the nuts and bushes, and a knife having the same radius as the specimen is forced through the specimen. For single shear tests the

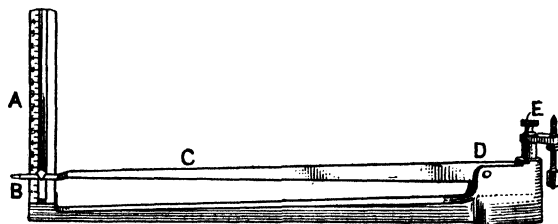



Fig. 126

specimen is slipped through one nut, and only allowed to project enough to give a full bearing on the knife.

162. **Deflection and Cross Breaking.** Tests of this type may be carried out in either the single lever or vertical screw

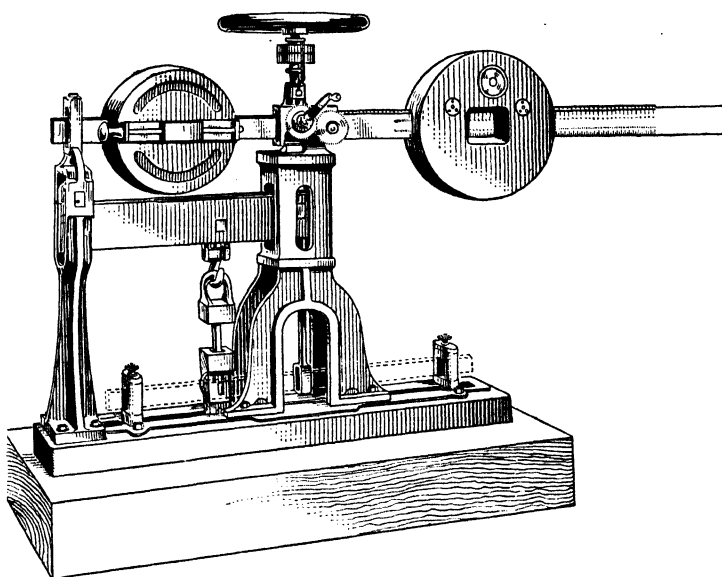



Fig. 127

machine. The specimen rests on knife edges (Fig. 106), and a knife edge, carried by the upper head, presses on the specimen at mid-span. Fig. 126 shows a deflection measuring device.

It consists of a cast-iron base, the bottom of which is accurately planed, and to which is attached a scale *A*, graduated in $\frac{1}{10}$ths inch. A multiplying lever *C* is pivoted on the base at *D*, and the beam presses on the projection *E* attached to the lever. The other end of the lever carries a vernier which enables readings to be taken to $\frac{1}{100}$ in. The scale *A* reads up to 5 in., and, as the multiplication obtained is 10 : 1, deflections can be read up to $\frac{1}{2}$ in.

For testing small bars in the foundry the machine shown in Fig. 127 has been designed by Messrs. W. and T. Avery, Ltd. It can also be used to carry out rough tensile tests, in which case a specimen of $\frac{1}{4}$ sq. in. cross-section is used.

For transverse tests, the test bar is inserted through adjustable dogs, the strain being applied by the hand-wheel, and the load is directly communicated to the steelyard. Thesteelyard is graduated up to 80 cwt. for tensile tests and 50 cwt. for transverse tests.

EXAMPLE 2.

A piece of oak, 1 in. wide and 2 in. deep and 40 in. long, was tested under bending by four-point loading, as indicated in Fig. 128. By this method of loading the central 24 in. of the beam is under uniform bending moment. The deflection was measured under various loads (until fracture occurred) on a length of 20 in. between points of application of the load, and the following observations were obtained—

Load in lb. (*W*)	0	100	300	500	700	900	1100	1300	1500	1700	1900	2100	2300	2500
Deflection on 20 in. in inches	0	0·014	0·042	0·071	0·100	0·129	0·157	0·192	0·233	0·283	0·340	0·410	0·500	0·650 beam broken

Plot the load-deflection curve and calculate, in pounds per square inch—
(*a*) The modulus of direct elasticity.
(*b*) The stress at the limit of proportionality.
(*c*) The modulus of rupture. (A.M.I.C.E., 1925.)

The load-deflection curve is shown plotted in Fig. 128, and on the same figure load and deflection are plotted on a larger scale in order to determine the limit of proportionality.

$$I = \frac{bd^3}{12} = \frac{2^3}{12} = \frac{2}{3} \text{ inch units.}$$

For a value of $W = CB = 950$ lb. the deflection is $AB = 0{\cdot}135$ in.

The uniform bending moment due to a total load $W = 950$ lb.

$$= \frac{950}{2} \times 6$$
$$= 2850 \text{ lb. in.}$$

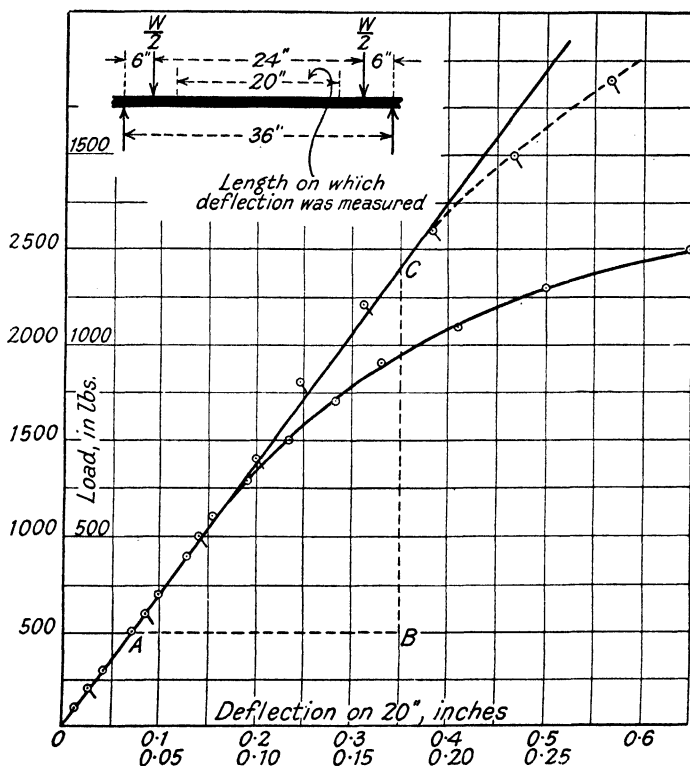



FIG. 128

The deflection due to a uniform bending moment is given by

$$Z = \frac{ML^2}{8EI} \text{ (par. 79 } (h))$$

$$\therefore \qquad E = \frac{ML^2}{8IZ}$$

$$= \frac{2850 \times 20 \times 20 \times 3}{8 \times 2 \times 0{\cdot}135}$$

$$= 1{\cdot}58 \times 10^6 \text{ lb./sq. in.}$$

The limit of proportionality occurs when $W = 1300$ lb.

$$M = \frac{1300}{2} \times 6 = 3900 \text{ lb. in.}$$

$$f = \frac{My}{I} = \frac{3900 \times 3}{2}$$

$$= 5850 \text{ lb. sq./in.}$$

In the testing of beams, the equation $\frac{M}{I} = \frac{f}{y}$ is not true beyond the limit of proportionality, nevertheless it is often *assumed* to hold up to the breaking-point, and by means of it a breaking stress calculated which is called the *Modulus of Rupture.*

Bending moment at instant of fracture

$$= \frac{2500}{2} \times 6 = 7500 \text{ lb. in.}$$

Modulus of rupture $= \frac{7500 \times 3}{2}$

$$= 11{,}250 \text{ lb. sq. in.}$$

163. **Torsion Testing.** A machine for testing materials in torsion is shown by Fig. 129. It is capable of dealing with specimens up to 1 in. diameter, and 12 in. in length. The specimen is held by a chuck at each end. One chuck is rigidly attached to a worm wheel, gearing with which is a worm whose spindle carries a hand wheel. The other chuck is connected through a lever and link to a graduated arm carrying a counterpoise. Rotation of the hand wheel causes rotation of the end of the specimen nearer the worm wheel; the other end of the specimen attempts to rotate and exerts a pull on the graduated arm. The counterpoise is now moved until the arm is floating in a horizontal position. Allowance is made for the shortening of the specimen due to torsion.

In order to measure the angular movement between two sections of the specimen a given distance apart, a disc graduated in degrees may be fastened to the specimen at one section, and one end of a long sleeve fastened at the other section. The other end of the sleeve carries a vernier which moves along the scale on the disc.

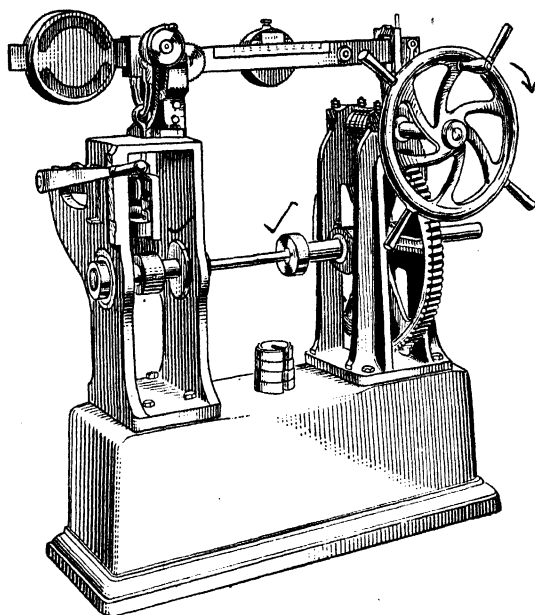



FIG. 129

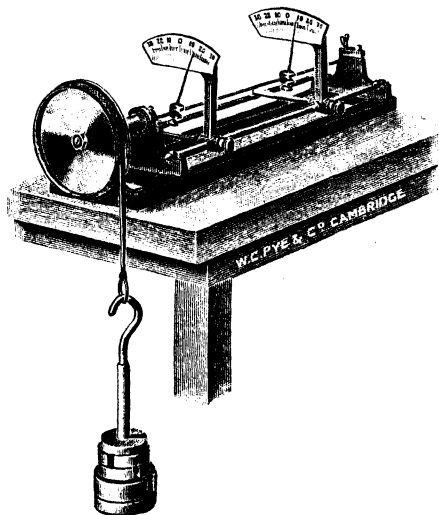



FIG. 130

For the testing of wires in torsion, the apparatus shown by Fig. 130 may be used. The wire under test is securely held at one end, and to the other end is attached a pulley, free to rotate. A cord passes round the pulley, and loads may be applied to a scale pan fastened to the free end of the cord. The relative angular movement of two sections of the wire is found by observing the readings of the pointers, attached to the wire on the graduated dials.

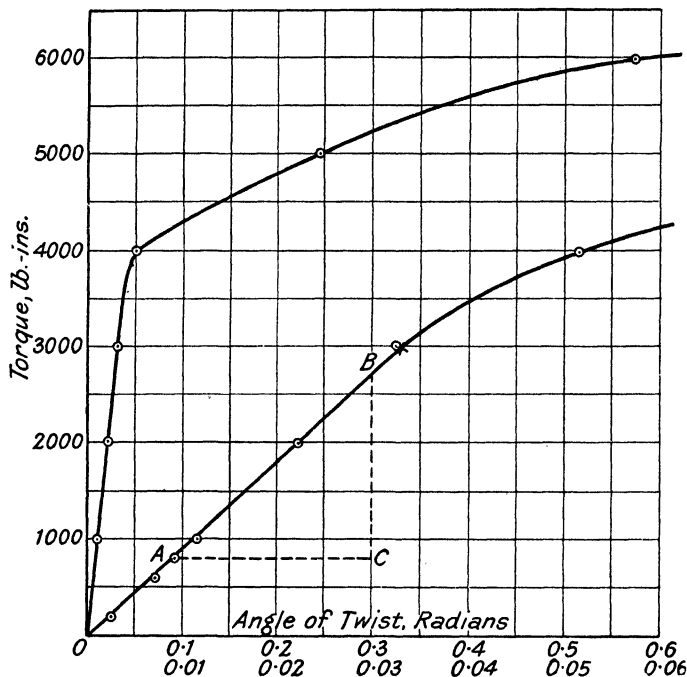



Fig. 131

Example 2.

Describe a method of determining the modulus of rigidity of a mild steel bar. A bar of steel 0·749 in. diameter, parallel length 4 in., was tested in torsion and gave the results shown in the table.

Torque (lb.-in.)	200	600	800	1,000	2,000	3,000	4,000	5,000	6,000
Angle of twist on length of 4 in. (radians)	·0026	·0071	·0092	·0116	·0223	·0325	·0517	·2453	·5779

The bar broke with a torque of 6,400 lb. in. Find (1) the modulus of rigidity, (2) the limit of proportionality, (3) the breaking stress in shear. (Lond. Univ., 1923.)

The angle of twist is shown plotted against torque in Fig. 131 for the complete range, and it will be observed that overstrain has occurred after about 4,000 lb. in. The quantities are plotted to a larger scale on the same figure in order to determine the modulus of rigidity and the limit of proportionality.

The torque BC = 1,900 lb. in. and the corresponding angle of twist = AC = 0·021 radians.

$$I_p = \frac{\pi r^4}{2} = \frac{\pi \times (0{\cdot}3745)^4}{2}$$

$$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$C = \frac{Tl}{I_p\theta} = \frac{1900 \times 4 \times 2}{\pi \times (0{\cdot}3745)^4 \times 0{\cdot}021}$$

$$= 11{\cdot}71 \times 10^6 \text{ lb./sq. in.}$$

The limit of proportionality occurs when the torque is 3,000 lb. in. and is given by

$$f = \frac{Tr}{I_p}$$

$$= \frac{3000 \times 0{\cdot}3745 \times 2}{\pi(0{\cdot}3745)^4}$$

$$= \frac{6000}{\pi(0{\cdot}3745)^3}$$

$$= 36{,}350 \text{ lb./sq. in.}$$

In testing ductile material to destruction, the material towards the end of overstrain becomes perfectly plastic, the overstrain curve becoming almost parallel to the axis. The shear stress is no longer proportional to the distance from the centre of the section, but is uniform in intensity over the section. The breaking torque may be found as follows—

Referring to Fig. 87

$$T = 2\pi f_b \int_0^r r_1^2 dr_1$$

$$= \tfrac{2}{3}\, \pi f_b r^3 \text{ where } f_b \text{ is the breaking shear stress}$$

$$\therefore \text{ Breaking stress} = \frac{\frac{3}{2} \times 6400}{\pi \times (0{\cdot}3745)^3}$$

$$= 58{,}160 \text{ lb./sq. in.}$$

164. Bending and Torsion. Fig. 132 shows diagrammatically a machine suitable for testing materials in combined bending and torsion. The test piece is fastened at one end to a graduated disc, capable of rotation by a worm and wheel, which is surrounded by a ring carrying a vernier. From the disc projects a rod carrying a scale pan to which loads W_1 may be applied. The other end of the test piece is fastened to a block A, suspended as shown. Projecting from the block, and in line with the test piece, is a rod carrying a scale pan to which loads W_2 may be applied. Rods also project from the

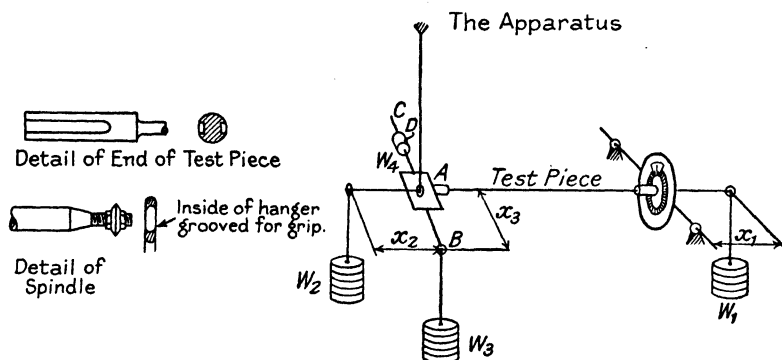



Fig. 132

block in a direction perpendicular to the axis of the test piece. One rod carries a scale pan to which loads W_3 are applied. The other rod carries a counterpoise D, which is used to balance the previous scale pan. The object of the test is to find the criterion of failure in a shaft subjected to bending and torsion. A uniform bending moment $W_1x_1 = W_2x_2$, is supplied by the weights W_1 and W_2, the twisting moment W_3x_3 is supplied by the load W_3. A spirit-level is attached to the block A. The bending moment is kept constant and loads W_3 applied at about 5 lb. at a time. After applying a load, the worm wheel is rotated until the spirit level again records zero, and the angle of twist noted. Twisting moment and angle of twist are plotted until departure from proportionality indicates that the elastic limit has been reached. Great care must be taken to prevent the bar from being permanently strained, W_3 being removed when an abnormal movement is noted on the spirit level. The bar may then be immersed for an hour in water

at 100° C., when it may be used again for a new test, or separate bars of the same material can be used. In each case a different value of bending moment is applied. The test piece is usually solid, ½ in. diameter, and 2 ft. long, or it may consist of a hollow tube.

If M = bending moment

T = twisting moment when elastic limit is reached,

we have from par. 115

From maximum stress theory

$$M + \sqrt{M^2 + T^2} \quad \text{should be constant} \qquad (1)$$

From maximum strain theory

$$\tfrac{3}{4}M + \tfrac{5}{4}\sqrt{M^2 + T^2} \quad \text{should be constant} \qquad (2)$$

From maximum shear stress theory

$$M\sqrt{^2 + T^2} \quad \text{should be constant} \qquad (3)$$

From maximum strain energy theory

$$\sqrt{\frac{2m}{m+1} M^2 + T^2} \quad \text{should be constant} \qquad (4)$$

Constancy of any of the above shows that failure occurs according to the particular theory from which the formula is derived. From the work carried out on various materials it would appear that a ductile material, such as mild steel, fails according to (3), whilst a brittle material, such as cast iron, fails according to (1).

165. Hardness Testing. The hardness of a body is the resistance which the body offers to indentation by other bodies.

The various tests may be divided into two classes, (*a*) Scratch tests, (*b*) Indentation tests.

The former usually consists of pressing a loaded diamond into the surface under test, then pulling the diamond in order to make a scratch. The hardness "number" is then based on the load required to make a scratch of given width, or else on the width of scratch made with a given load.

Indentation tests usually consist in pressing a body of standard shape into the material to be tested under a given load, and basing the hardness number on the load and the indent produced.

Another instrument, called a scleroscope, has been designed to measure the rebound of a small hammer, which carries a diamond for indenting purposes. The hammer slides inside

a glass tube from a given height. and the height of the rebound is a measure of the hardness.

These methods of testing are open to the following objections—

(1) In each case the hardness is measured relative to a standard substance, and it is difficult to get the standard substances all of the same degree of hardness.

(2) The standard substance will be slightly deformed, due to the force exerted, and hence in the case of indentation tests a true value of the indent is not obtained.

(3) The material under test will rise in the form of a ridge around the indenting tool, and thus prevent a true value of indent being obtained.

(4) The material under test, although stressed beyond the yield point at the point of application of the indenting tool possesses some residual elasticity and thus, on removal of the tool, the indent will recover slightly, so that an erroneous value of indent will be obtained.

The indentation method is the commonest form of test, and machines designed on this principle will now be described.

166. **The Rockwell Method of Hardness Testing.** In this machine an indenting tool which is either a steel ball or a conical shaped diamond with a rounded point, called a "brale," is driven into the material to be tested, under a given load. The load and tool used depend on the material under test. A ball $\frac{1}{16}$ in. in diameter and a load of 100 kgs., together with Scale B on the instrument dial, is used for such materials as low and medium carbon steels. For harder steels, the brale with a 150 kg. load is usual, and the dial marked C is employed. For soft materials balls of diameter varying from $\frac{1}{8}$ to $\frac{1}{4}$ or $\frac{1}{2}$ in. may be employed with special loads and dial graduations. Very thin material can be tested by this method. The brale, whose cone angle is 120°, or the steel ball, is made to carry a minor load of 10 kgs. The full load is then applied and, after removal, leaving the minor load still in position, the hardness is read on the proper scale.

A special adaptation of the machine called a superficial hardness tester has been devised for testing nitrided steel, safety razor blades, lightly carburized material, thin sheets, etc. The loads are reduced to 15, 30, or 45 kgs., the brale being employed for the nitrided steels and the $\frac{1}{16}$ in. diameter ball on thin sheets.

167. **Brinell's Method of Hardness Testing.** This method consists of squeezing a hardened steel ball of given diameter into the material, under a fixed load. The load is usually

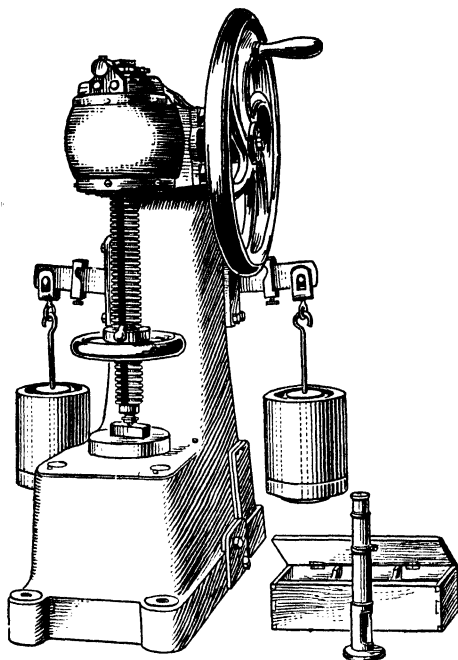



Fig. 133

3,000 kg. and the ball has a diameter of 10 mm. Smaller loads are used for the softer materials.

Let P = standard load in kilograms

D = diameter of ball in millimetres

d = diameter of indent in millimetres

$$\text{Brinell hardness number} = \frac{\text{load in kgs.}}{\text{spherical area of indent in sq. mm.}}$$

$$= \frac{P}{\frac{\pi D}{2}\left(D - \sqrt{D^2 - d^2}\right)}$$

This method of hardness testing is more widely used than any of the others. The machines designed on this principle depend

either on hydraulic pressure where the force is measured by a gauge, or on a powerful screw, and the force in this case is measured by a steelyard connected through levers to the table carrying the specimen.

A machine of the latter type is shown in Fig. 133; the specimen is placed on the table and the screw carrying the ball is lowered until the ball rests on the specimen. The screw at

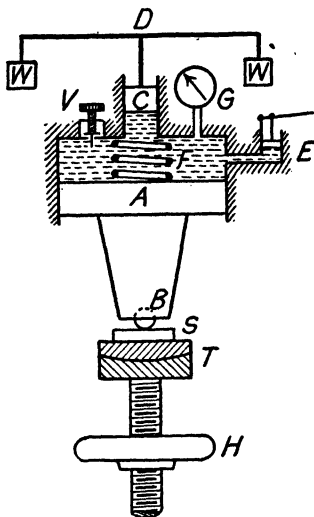



FIG. 134

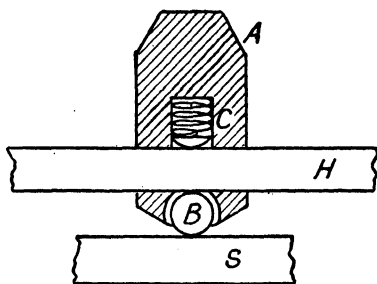



FIG. 135

its upper end is attached to a worm, in gear with which is a worm wheel carrying a hand wheel. Rotation of the hand wheel causes the screw to advance and press on the specimen. The table of the machine is connected through levers to the two arms carrying weights which project from each side of the machine. When the desired pressure is reached the weights are elevated. The microscope is used to measure the diameter of the indent.

A machine operated by hydraulic pressure is shown diagrammatically in Fig. 134.

In the case of a machine part which is too large to be placed in either of the machines just described, an instrument called a hardness comparator (Fig. 135) may be used. This instrument is of a size which slips into a pocket and is thus useful for testing the hardness of an object which it is not desirable to move.

It consists of a steel cylinder A, carrying a ball B resting on the specimen and also touching a bar H of standard Brinell hardness, which can be inserted in A. A spring C holds H in position. The top of the cylinder is struck with a hammer and an indent is made both in the bar and in the specimen.

Let P = force of the blow.

A_1 = area of indent in standard bar.

A_2 = ,, ,, ,, ,, specimen.

B_1 = Brinell hardness number for bar.

B_2 = ,, ,, ,, for material.

$$P = k\,(\text{Brinell No.} \times \text{area of indent}) = kA_1B_1 = kA_2B$$

$$\therefore\; B_2 = \frac{A_1B_1}{A_2}$$

167A. The Vickers or Diamond Hardness Method of Testing. In machines of this type the hardness number is determined by pressing a square-based diamond pyramid into the material. The angle between opposite faces is 136° and the load may vary from 5 to 120 kgs. The hardness number is the ratio P/A where P is the load in kgs. and A is the surface area of the indentation. It is claimed that the diagonal of the square indent can be measured with greater accuracy than the diameter of a circle. The method can be used for exceedingly thin materials and for very hard materials. On p. 387 is given a table which shows the approximate corresponding values of hardness, as found by such a machine, the Brinell and Rockwell machines and the Shore Scleroscope. The table has been prepared from tests carried out on ordinary carbon and alloy steels and *should be applied with care to high alloy austenitic steels and non-ferrous alloys* in which the relationship between hardness number and tensile strength in particular is somewhat different from that of ordinary steel.

No fundamental basis of comparison exists between the hardness numbers as determined by different methods because the methods do not in all cases measure the same property of the metal. It is hoped, however, that the table will be useful in making a general comparison between tests carried out by different methods.

168. **Connection between Brinell Hardness and Strength.** For the same quality of material a relationship seems to exist between the ultimate strength and the Brinell hardness number. In the case of the family of steels given in table in par. 150, the ultimate strength divided by the hardness number varies from 0·21 to 0·23, and in the case of the steel given in par. 153, tempered at different temperatures, the ultimate strength divided by the hardness number varies from 0·215 to 0·235.

169. **Impact Testing.** The ordinary static tensile test is not satisfactory when we wish to determine the suitability of a material to resist repeated shocks. Machines have been devised in which a specimen receives a single shock or a number of shocks, and the energy required to smash the specimen is taken as a measure of the resistance of the material to this manner of stressing.

The various testing machines may be divided roughly into two classes : (*a*) those in which a notched specimen receives a single transverse shock, or a number of transverse shocks ; and (*b*) those in which the specimen receives a single tensile or compressive shock, or a series of shocks of similar type.

The machine shown by Fig. 136 is of the single transverse shock type. It consists of a heavy pendulum pivoted at the top of two strong A-frames. A pointer attached to the pendulum moves over a scale graduated in foot-pounds, and mounted on top of the machine. The specimen, which consists of a piece of material of square section 10 mm. side, is notched in one face. The notch is 2 mm. deep, and has a radius at the bottom of 0·25 mm. ; a notch-measuring gauge being supplied so that specimens can be accurately machined to size. The specimen is held in a vice, fastened to the base plate of the machine ; the notch being at the top of the vice jaw and facing the pendulum. The pendulum is lifted into position and secured, and the energy stored up in it is 120 ft. lb. On being released by a trigger, the specimen is struck and broken, and the residual energy in the pendulum is recorded on the graduated scale. The difference between the energy stored before release of the pendulum and the residual energy is taken as the energy required to smash the specimen. A test of this kind is known as the " Izod Test," and is widely used to measure the resistance of materials which are to be subjected to shock.

A machine for delivering a succession of transverse blows to a specimen has been designed by Dr. Stanton. In this

machine a hammer is allowed to fall repeatedly on the specimen from a given height, which height can be varied. The specimen is supported horizontally at each end, is cylindrical, and of

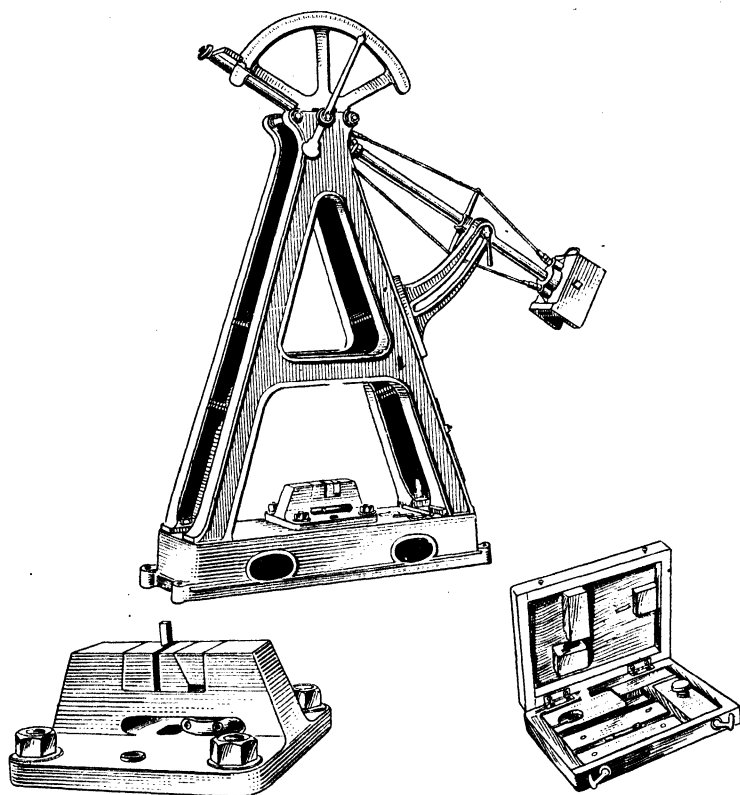



FIG. 136

$\frac{1}{2}$ in. diameter. A V-groove is cut in the specimen at mid-span, at which point the transverse blow is struck. Arrangement is made to rotate the specimen through 180° between successive blows. The machine is capable of delivering 100 blows per minute and is automatically stopped when the specimen is fractured ; the number of blows being recorded on a revolution counter. The weight of the hammer × height fallen × number of blows = energy absorbed by specimen.

In machines of the type just described it is not correct to

assume that the energy as stated is the energy required to rupture the specimen, since portion of the available energy is utilized in straining the hammer and the specimen at the point where the blow is struck. This energy cannot be calculated, and thus it is impossible to calculate the exact amount of energy absorbed by the specimen.

A machine has been designed by Sankey, Blount and Kirkcaldy* in which the specimen is subjected to a tensile shock applied in such manner that the energy absorbed by the specimen can be computed. The principle underlying the machine is shown in Fig. 137. The specimen S is fastened to a tup W, and to a head H, and the whole arrangement is lifted to a height h, at which the potential energy of the tup is Wh. The arrangement is released and falls freely, the head H being arrested by the anvil A. The height h is so chosen that the specimen is fractured. An electrical timing device enables the velocity of the tup to be calculated *after* fracture and hence its kinetic energy E could be estimated.

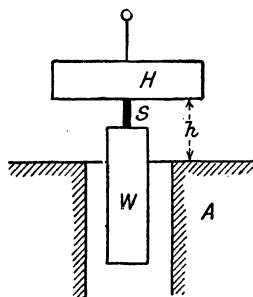



FIG. 137

The energy absorbed by specimen $= Wh - E$.

This machine was of the "single shock" type, and, after investigating various steels, the experimenters concluded that the energy required to fracture a specimen by a single tensile shock was 50 per cent greater than when fracture was carried out by the static method.

The author† has designed a machine (Fig. 138) working on the same principle, but the machine may be used for single shock tests or repeated shock tests.

The machine consists of a hollow, cast-iron anvil carrying a heavy hollow steel top. The specimen Q is attached to a tup B and an arresting head A; this head carries cross-guide arms, but sufficient clearance is given to allow of a *free* fall. A sliding head C carries two triggers which are capable of opening against the pressure of a spring and grasping a knife-edge on top of the arresting head A. The head C slides on two vertical rods D. A cross-bar E carries adjustable releasing plates. The sliding head C is attached to a rope which passes

* *Proc. Inst. Mech. Eng.*, May, 1910.
† *Proc. Iron and Steel Inst.*, Oct., 1927.

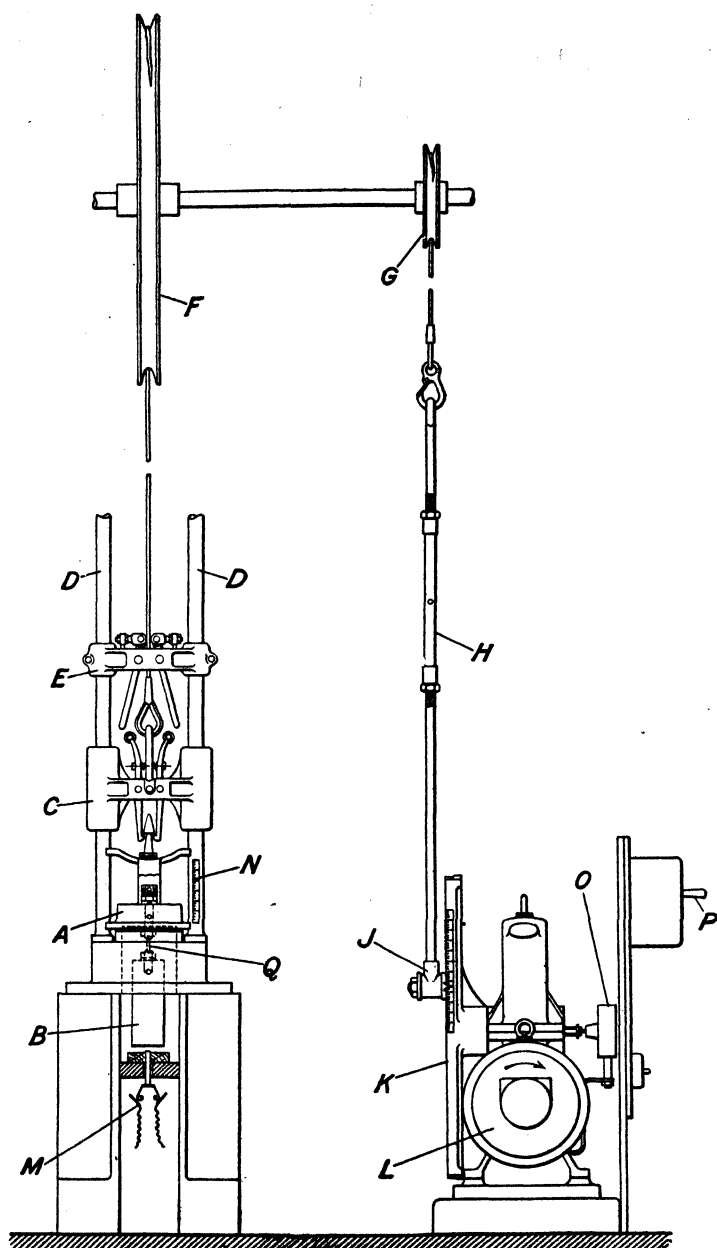



FIG. 138

through E and is fastened to the rim of a wheel F. On the same shaft as F is a small wheel G, to which is attached a rope, the other end of which is attached to a connecting-rod,

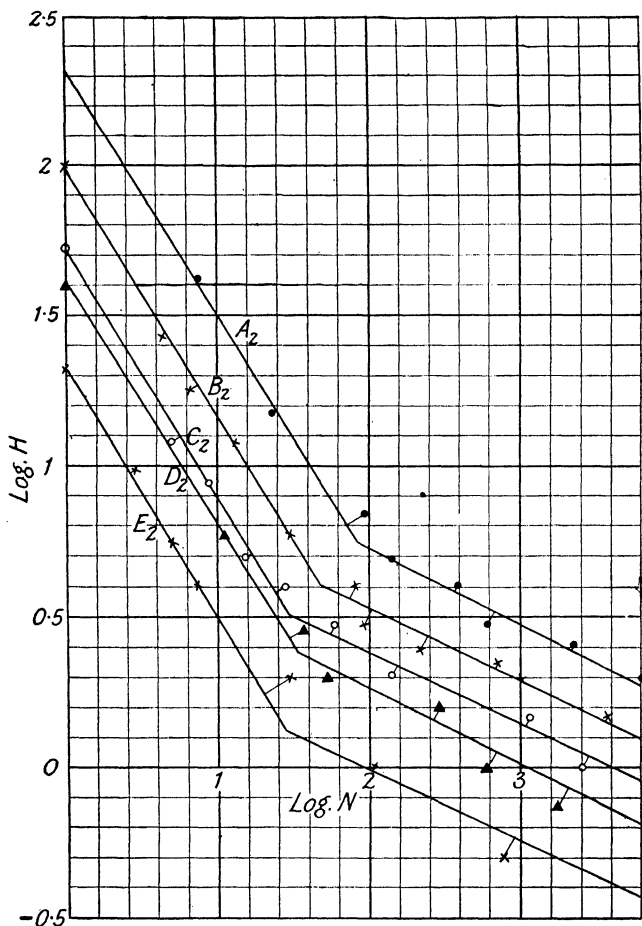



FIG. 139

which, in turn, is connected to the crank K by the adjustable pin J. The crank K is attached to a reducing gear driven by the motor L. There is a ratio of 8 to 1 in F and G, thus the crank-pin radius for a given fall is obtained by dividing the fall by eight. Having adjusted the pin J, the turnbuckle

H is adjusted so that the zero position of lift is at the anvil. The motor is rotated by hand until C is at the top of the lift and E is then set to release the head A by squeezing in the triggers. The motor is now started, and the head A, specimen Q, and tup B are automatically lifted, released, and arrested, until the specimen fractures. The tup falling on M causes a cut-out to operate and stop the motor; the number of shocks being recorded on counter O.

The following is a description of some of the experiments carried out on this machine: Specimens are broken with a given tup falling repeatedly from a given height, the height being different for each specimen. The value of the product of height of fall, number of shocks and weight of tup give the energy absorbed by the specimen. The values of log. height were plotted against the log. of the number of shocks, and curves such as those shown in Fig. 139 were obtained. These are for specimens of Low Moor iron, A_2, B_2, etc., representing tups of 2, 4, 7, 10 and 20 lb. respectively. The point where each cuts the axis of log. H gives the log. of the height through which the fall must take place with each tup in order to fracture the specimen with a single blow. The value of the product of height of fall and weight of tup was found to be constant. Specimens were broken by static means and the work done up to fracture calculated.

The author finds that the ratio $\dfrac{\text{single shock energy}}{\text{static energy}} = 1.$

The heights of fall, for fracture by a single shock, as found by the above method, were tested by allowing a specimen, the tup, and arresting head to fall from a height 1 in. below that as found; then allowing another specimen, etc., to fall from a height 1 in. above the height as found from the graph. In the former case fracture did not occur, whilst in the latter case fracture did occur.

Stanton and Bairstow* give results of a number of single blow impact tests carried out by them. They obtained a ratio of $\dfrac{\text{impact energy}}{\text{static energy}} = 1{\cdot}5$. In their machine, however, a great amount of energy is dissipated in straining the hammer, etc., and they give this as the reason for the value of the ratio

* *Proc. Inst. Mech. Eng.*, 1908.

obtained from their tests. Materials tested by repeated impact show the usual "fatigue" curves associated with varying or alternating stress.

170. **The Fatigue of Metals.** Metals which are subjected to loads which vary continuously from one value to another, or from one kind of loading to the reverse kind, fracture at values of stress much below the ultimate stress as found by static tests.

This peculiarity of metals was first subjected to an extended investigation by Wöhler, and since then many experimenters have carried on the investigation. The cause of failure is still imperfectly understood, and a complete history of the work carried out would fill many volumes. The reader is advised to apply to the proceedings of the various engineering societies, or to the *Fatigue of Metals*, by H. J. Gough,* for a full account of the work carried out. A synopsis of the more important results of the above investigations will now be given.

From Wöhler's experiments the following results were obtained—

(*a*) That reversed stresses, much smaller in magnitude than the static breaking stress, and even smaller than the elastic limit, will cause fracture if repeated a sufficient number of times.

(*b*) Under fluctuating stress the resistance to fracture depends upon the range of stress within certain limits.

The range of stress is given by $R = f_{max} - f_{min}$

where R denotes range of stress,

f_{max} denotes maximum stress,

f_{min} denotes minimum stress,

due attention being given to the positive and negative values of the stresses.

These results have been verified by other investigators, and the following tables† will serve to illustrate them. All stresses are measured in tons/sq. in.

The steel used for Table I is the 0·34 carbon steel of par 150, and that in Table II is the 0·19 carbon steel of the same paragraph. Reference to this paragraph shows us that 0·19 carbon steel had a yield stress of 17·19 tons/sq. in. and a maximum

* Scott, Greenwood & Co.

† "Some Experiments on the Fatigue of Metals," J. H. Smith, *Journal of Iron and Steel Inst.*, No. II for 1910.

TABLE I

RANGE OF STRESS AND ENDURANCE

Maximum Stress	Minimum Stress	Range of Stress	Alternations of Stress
19·7	- 19·7	39·4	9,450
17·6	- 17·6	35·2	25,600
21·9	- 0·3	22·2	449,260
- 8·7	- 24·3	15·6	1,461,500
26·9	14·0	12·9	2,750,000 (not broken)

TABLE II

REVERSED STRESS AND ENDURANCE

Maximum Stress	Minimum Stress	Range of Stress	Alternations of Stress
19·8	- 19·8	39·7	1,500
18·1	- 18·1	36·2	10,650
17·1	- 17·1	34·2	22,400
16·8	- 16·8	33·6	38,600
14·9	- 14·9	29·9	127,600
13·5	- 13·5	27·0	490,000
12·6	- 12·6	25·2	739,400
12·3	- 12·3	24·6	1,066,000 (not broken)

stress of 29 tons/sq. in. when tested statically ; from Table II, however, it is shown that a much smaller reversed stress than the yield stress is sufficient to cause failure when applied a sufficient number of times.

The values of range of stress given in Table I are shown plotted against the corresponding alternations of stress in Fig. 140. It will be observed that as the number of alternations increase the curve becomes almost parallel to the axis, corresponding to a range of stress of 12·9 tons/sq. in. This range of stress, at which the number of alternations required to cause fracture becomes almost infinite, is called the "Limiting Range of Stress."

171. **Bauschinger's Experiments.** When a material is tested statically, in tension or compression, without receiving further treatment than that obtained during manufacture, it shows clearly defined limits of proportionality. Treating these as elastic limits, it is usual to apply to them the term " primitive elastic limits."

Bauschinger showed that these primitive elastic limits were not fixed, but that when the material was subjected to cycles of stress new limits of elasticity could be produced, the limits being either raised or lowered, but not necessarily by equal amounts. These new limits are called "natural elastic limits." Bairstow* demonstrated the truth of the above statement and

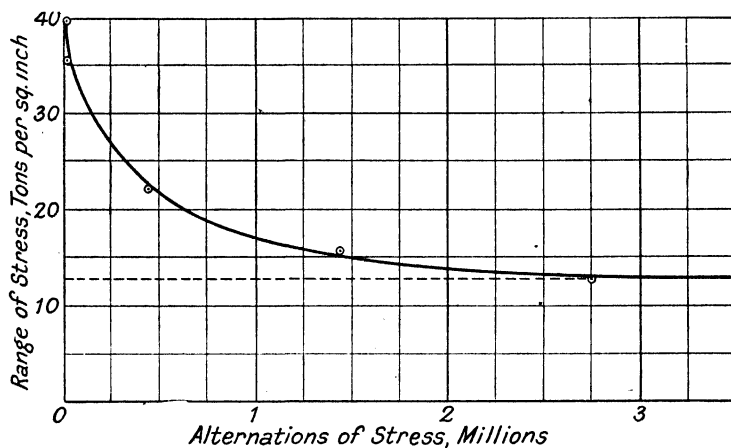



FIG. 140

also was led to the conclusion that the *fatigue range was an elastic range.*

172. "**Strain**" **Method of obtaining Fatigue Ranges.** Experiments of the type carried out by Wöhler require an exceedingly long period of time as well as a large number of specimens in order to determine the fatigue range of a material. A "strain" method has been invented by Dr. J. H. Smith† which occupies a relatively short space of time and allows the fatigue range to be obtained with a *single specimen* without damaging the specimen. He attached an extensometer of the optical type to the specimen and obtained a continuous band of light on a ground-glass scale. The length of this band was a measure of the cyclic strain of the specimen. The stress range was varied by increasing the speed of the machine (par. 176) and noting the length of the line. The process was repeated with increased stress ranges. On plotting length of line and

* *Phil. Trans. Roy. Soc.*, Vol. A, 1910.
† *Journal of Iron and Steel Inst.*, No. II, Vol. 1910.

stress range (Fig. 141) it is found that proportionality holds up to a certain point at which a definite " yield " is observed. After comparing the limiting ranges as found by each method, Smith concluded that the limiting range of stress, or fatigue range, was identical with the "yield range" as found by the strain method.

173. **Variation of Limiting Range of Stress.** (*a*) *Effect of Change of Section.* An abrupt change of section causes a

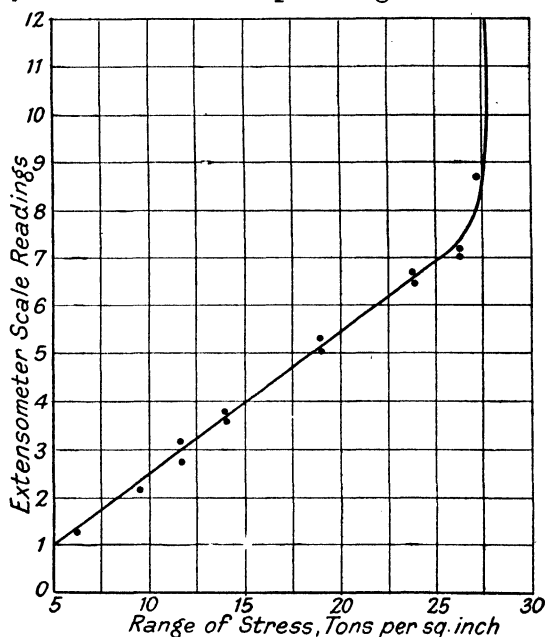



FIG. 141

serious reduction in the safe range of stress ; this point is clearly brought out by referring to Fig. 122 and the accompanying table in par. 158. Thus it is of the utmost importance that, in all machine parts subjected to repeated stresses, care should be taken to avoid sudden changes of section. The reduction is greater for the harder steels than for the more ductile steels.

(*b*) *Surface Finish.* It has been pointed out by Eden, Rose and Cunningham* that fineness of surface finish has a decided

* *Proc. Inst. Mech. Eng.*, 1911.

effect on the fatigue range. Other investigators have verified this, and in general the finer the surface finish the greater will be the fatigue range. From the point of view of design it is therefore of importance that all parts subjected to fluctuating stresses should have as good a surface finish as possible, and everything in the nature of a scratch should be avoided.

(*c*) *Under-stressing.* If a material is subjected to repetitions of a safe range of stress and then tested at a higher range of stress, it will be found that the material has been strengthened. This operation is known as "under-stressing."

(*d*) *Over-stressing.* If a material is subjected to fluctuating ranges of stress greater than the fatigue range, it is said to be "over-stressed." The investigations on over-stressing tend to show that for any degree of over-stress there is a limiting number of repetitions below which the fatigue range is unaffected, also this limiting number of repetitions decreases as the amount of over-stress increases. A large percentage of over-stress causes a decided weakening of the material. The effect of over-stress is more marked on the harder steels than on the softer steels.

(*e*) *Speed of Repetition.* The effect of various speeds of operation on the fatigue range of different materials has been investigated by numerous experimenters, the speeds varying in some cases from 200 to 5,000 reversals per minute. It is now the accepted theory that speed of repetition has no effect on the fatigue range.

174. **Hysteresis Loops.** Experimenting with long wires, which were loaded and unloaded, Ewing* showed that, with stresses well below the elastic limit, the strain during the loading stage was smaller than the strain during the unloading stage for equal values of stress. This phenomenōn has been referred to in par. 157, and was called hysteresis.

Let a specimen be gradually loaded in tension, then unloaded, and a gradually increasing compressive load added of magnitude equal to that of the tensile load. Upon removal of the compressive load and re-loading to the original value of tension, a loop is obtained when stress and strain are plotted.

If the process is repeated continuously, the same loop will be traced out; the material is now said to be in the "cyclic state," and its elastic stresses are equal in tension and compression. The loop obtained by the above process is called a

* *Brit. Assoc. Rep.*, 1899.

"hysteresis loop" for the given stress range. Such a loop is shown by Fig. 142.

If the range of stress is increased, the width of the hysteresis loop also increases, and consequently gives an increased area enclosed by the loop. When the stress range is decreased the width and area of the loop also decreases until it finally becomes

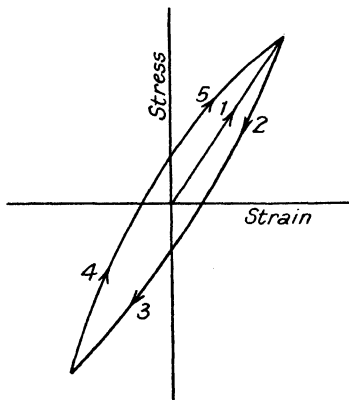



FIG. 142

a straight line. The range of stress in this case appears to be the same as the Bauschinger range.

The behaviour of these loops has been investigated by Smith and Wedgewood* and the reader is referred to their investigation for full information.

175. **Formulae connecting Stress Range, Maximum Stress, and Ultimate Strength.** Various empirical formulae have been devised to connect the above quantities. The following are the best known and are fairly representative—

Let f_{max} = maximum stress
R = stress range
f = ultimate strength
f_{min} = minimum stress

(*a*) *Gerber's Formula.*

$$f_{max} = \frac{R}{2} + \sqrt{f^2 - nRf}$$

where n is an experimental constant for the material.

* "Stress Strain Loops for Steel in the Cyclic State" (*Journal Iron and Steel Inst.*, 1915).

(*b*) *Launhardt's Formula.*

This formula applies only to stress ranges whose limits are both of the same type of stress. It is expressed as follows—

$$f_{max} = A + \frac{f_{min}}{f_{max}}(f - A)$$

where A is the safe stress range when one of the limits of stress is zero, this value being obtained experimentally.

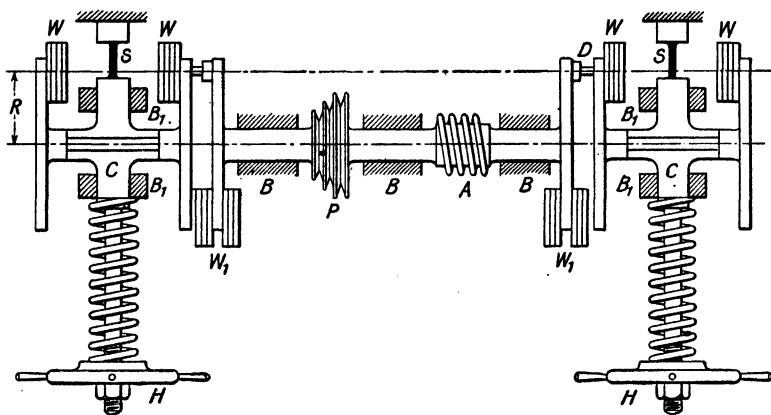



FIG. 143

(*c*) *Weyrauch's Formula.*

$$f_{max} = A - \frac{f_{min}}{f_{max}}\left(A - \frac{R_r}{2}\right)$$

where R_r is the safe range of stress for reversed stresses. The formula is applicable to reversed stress cycles and cycles varying from zero to a maximum.

176. **Testing Machine for Repeated Stresses.** Many machines have been devised for investigating the properties of metals when subjected to repeated stresses. The more successful investigators have produced machines in which the stresses vary in a simple harmonic manner. The machine shown graphically by Fig. 143 is that invented by Dr. J. H. Smith, a full description of which will be found in *Engineering*, 23rd July, 1909.

The machine is designed to take two specimens S, each of which is rigidly attached at one end to the frame of the machine.

The other end of each specimen is attached to the members C, which act as a bearing. These members have rotating shafts which have cranks mounted at each end carrying the masses W. The cranks are connected by special couplings D to cranks on each end of the driving shaft. Each of these cranks carries balance weights W_1. A is a worm which operates an oil pump and tachometer, and P is the driving pulley ; bearings are provided at B and B_1. The coupling D allows free movement of the masses, cranks, and members C, in the direction of the axis of the specimen.

Due to the centrifugal forces exerted by the masses W, reversed direct stresses are applied to the specimens. Any desired stress can be superimposed on the alternating stresses by means of the spring and hand wheels H.

Let a = area of cross-section of specimen

F = load imposed by spring

Then the fluctuating stresses are given by

$$f = \left(F \pm \frac{2W}{g}\,\omega^2 R\right)\frac{1}{a}$$

where ω = angular velocity of the cranks.

The machine can be uncoupled readily at one of the couplings D, and run as a single unit if it is not required to test two specimens simultaneously. The motive power is supplied by a variable speed motor with a rope drive. The usual running speed is 1,000 r.p.m.

177. **Cement Testing.** It is not usual in engineering practice to employ cement alone or "neat." It is usually mixed with broken stone, etc.; also it is rarely employed to take tensile stress, but is widely made use of to take thrust. The tensile test of cement is widely used, however, as it is found to give a good indication of the quality of the cement. The Engineering Standards Committee have laid down the following standard tests for cement.

(*a*) *Fineness Test.* If the cement is coarse, its strength is lowered when set, and the following are the conditions of fineness—

100 grammes shall be continuously sifted for 15 min. on a B.S. Mesh No. 170, and the residue on a B.S. Mesh No. 72 for 5 min.

The residue on a sieve of B.S. Mesh No. 72 shall not exceed 1 per cent, and the residue on a sieve B.S. Mesh No. 170 shall not exceed 10 per cent.

(*b*) *Chemical Composition.* The proportion of lime, after deduction of the proportion necessary to combine with the sulphuric anhydride present, to silica and alumina when calculated (in chemical equivalents) by the formula

$$\frac{CaO}{Si\,O_2 + Al_2\,O_3}$$

shall not be greater than 3·0 nor less than 2·0. The percentage of insoluble residue shall not exceed 1·0 per cent; that of magnesia shall not exceed 4 per cent, and the total sulphur content calculated as sulphuric anhydride (SO_3) shall not exceed 2·75 per cent. The total loss on ignition shall not exceed 3 per cent for cement manufactured or sampled or tested in temperate climates and 4 per cent for cement manufactured or sampled or tested in hot climates.

(*c*) *Consistency.* Cement paste of normal consistency shall be used in the tests for soundness, setting and tensile strength. For the purpose of arriving at this consistency, the Vicat apparatus, Fig. 145, shall be used, the needle being replaced by a plunger 1 cm. diameter. The quantity of water to be added to produce a normal paste shall be 0·78 of that required to give a paste which will permit the settlement of the Vicat plunger to a point 5 to 7 mm. from the bottom of the mould. The time from adding the water until commencing to fill the mould shall not be less than 3 min. nor more than 5 min. For quick-setting cement the time shall not be less than 2 min. nor more than 3 min., and the mould shall be completed in 5 min.

Trial pastes shall be made up of varying percentages of water until the normal consistency is reached, and the amount of water for this is then recorded and expressed as a percentage by weight of dry cement.

(*d*) *Soundness Test.* This test is carried out in order to determine the amount of free lime in the cement, and is known as the "Le Chatelier" test. The apparatus shown by Fig. 144 consists of a small split brass cylinder of thickness 0·5 mm. It forms a mould 30 mm. high and 30 mm. diameter. Pointers are attached to the cylinder on each side of the split. These pointers are 165 mm. in length from the tip to the centre of the

cylinder. A plastic mixture of cement and water as prepared for normal consistency is placed in the mould, which rests on a glass plate, another glass plate being placed on top of the mould and weighted. The whole apparatus is then placed in a water bath whose temperature is between 58 and 64° F. The distance between the tips of the pointers is measured after an interval of twenty-four hours. The mould and cement are now immersed in cold water, which is heated to boiling point

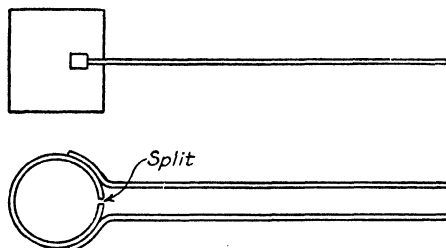



FIG. 144

in 25 to 30 min. and kept at this temperature for three hours. After cooling, the distance between the tips of the pointers is again measured. The expansion must not exceed 5 mm. for a cement which has been exposed to the atmosphere for seven days before mixing, or 10 mm. if unexposed.

(*e*) *Time of Setting.* This test is usually carried out by means of a weighted needle as shown in Fig. 145. A pat of the cement which has been mixed as for normal consistency into a plastic state is held in a circular mould. The needle, which is 1 mm. square, is gently lowered into the surface of the cement. When the needle fails to completely penetrate the pat the cement is said to be "set." This is called the initial set, and the time is that from adding the water until the needle ceases to penetrate the pat.

For the determination of final set, the needle is replaced by one having a diameter of 5 mm.; final set occurs when no impression is made by this needle, but a slight impression may be made by the former one.

Quick-setting cement will be set initially in not less than 5 min., finally in not more than 30 min.

Medium-setting cement will be set initially in not less than 30 min., finally in not less than 10 hours.

The needle and rod to which it is attached weigh 300 grm. A pointer attached to the rod slides on a scale and gives the depth of impression at any given time.

(*f*) *Tensile Test.* The cement is mixed as for normal consistency, with sufficient water to make it plastic. It is then put into a standard mould, whose dimensions are given by Fig. 146, without being rammed mechanically. The mould is 1 in. deep and thus the minimum cross-section of the briquette will have an area of 1 sq. in. The mould rests on a steel plate, and as soon as the setting of the briquette will permit the mould is removed. The briquette shall be kept in an atmosphere of at least 90 per cent relative humidity for twenty-four hours, after which it is placed in a bath of fresh water, of temperature 60° F., until required for testing purposes. The water is changed every seventh day.

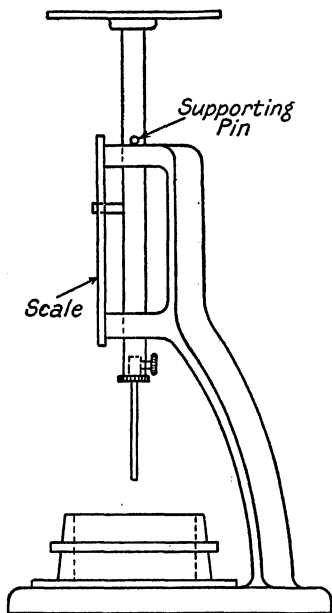
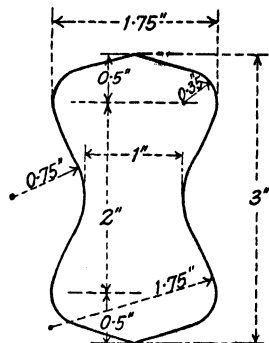



FIG. 145

FIG. 146

It is usual to prepare twelve briquettes, six of which are tested after three days, and six after seven days in the case of cement and sand. The following are the minimum tensile strengths laid down by the Engineering Standards Committee.

Neat Test.

After 7 days . . 600 lb./sq. in.

The tensile test for "Neat" cement is optional and is not required by the Engineering Standards Committee.

(*g*) *Sand Test.* The standard sand is obtained from Leighton Buzzard, and the mixture consists of one part by weight of cement to three parts by weight of sand. The percentage water to be used being determined from: $\frac{1}{4} P + 2{\cdot}5$, where P is the percentage of water to produce a paste of normal consistency.

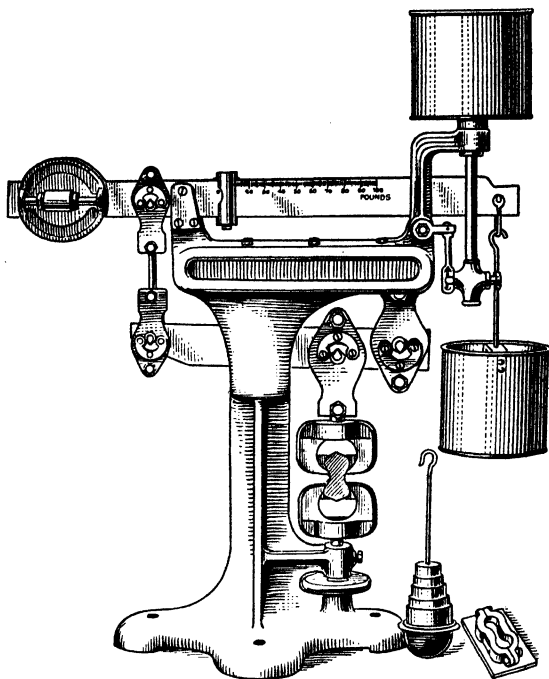



Fig. 147

The briquettes are treated as in the "Neat" test. The minimum tensile strength should be

After 3 days . .	not less than 300 lb./sq. in.
After 7 ,, . .	not less than 375 lb./sq. in.

178. **Cement Testing Machine.** The machine shown in Fig. 147 is of the compound lever type. The briquette is gripped in the specially shaped jaws, which are well greased, and sufficient tension applied by the hand-wheel to raise the steelyard to the top of its movement. Lead shot is allowed to

flow from the upper receptacle into the pan suspended from the end of the steelyard, thus causing tensile stress in the specimen. When fracture occurs, the steelyard falls and auto matically cuts off the supply of shot. The shot in the pan is then weighed on the steelyard by moving the counterpoise until balance is obtained, the steelyard being graduated in terms of breaking load.

The rate of application of the load affects the breaking load, the load increasing with the speed of loading, and it is laid

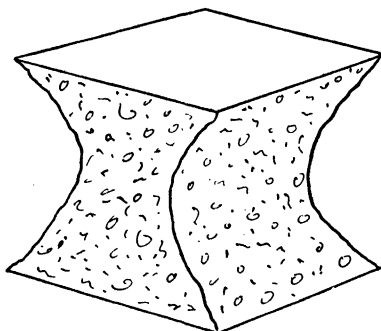



Fig. 148

down that the load shall be applied at a standard rate of 100 lb. per sq. in. of section in 12 sec. The orifice through which the lead falls has been designed to give this rate of flow.

179. **Testing of Concrete, Stone, Brick, etc.** Concrete and stone are usually tested in the form of cubes of 4, 6, 8, and 12 in. edge, but cylinders are also used. In the case of concrete the strength increases with age and depends largely on the mixture.

The strength of bricks is affected by the manufacturing process and by the composition of the clay.

It is of the utmost importance in testing these brittle materials that a good bedding surface should be obtained, in order that the pressure may be evenly distributed over the surface. Lead plates are sometimes used above and below the cubes or cylinders, but better results are obtained when a thin coating of plaster of paris is applied to the surfaces taking the thrust, in order to make them smooth and parallel, and millboards take the place of the lead plates. The compression blocks of the testing machine should have spherical seatings and the cubes should be placed centrally on these.

Fracture in concrete takes place as shown by Fig. 148, at angles of about 45° to the direction of thrust; this form of fracture being common to other brittle materials as shown by Fig. 124 (*b*).

180. **Lamb's Extensometer.** The ingenious extensometer* designed by Prof. E. H. Lamb is of the optical type. It can

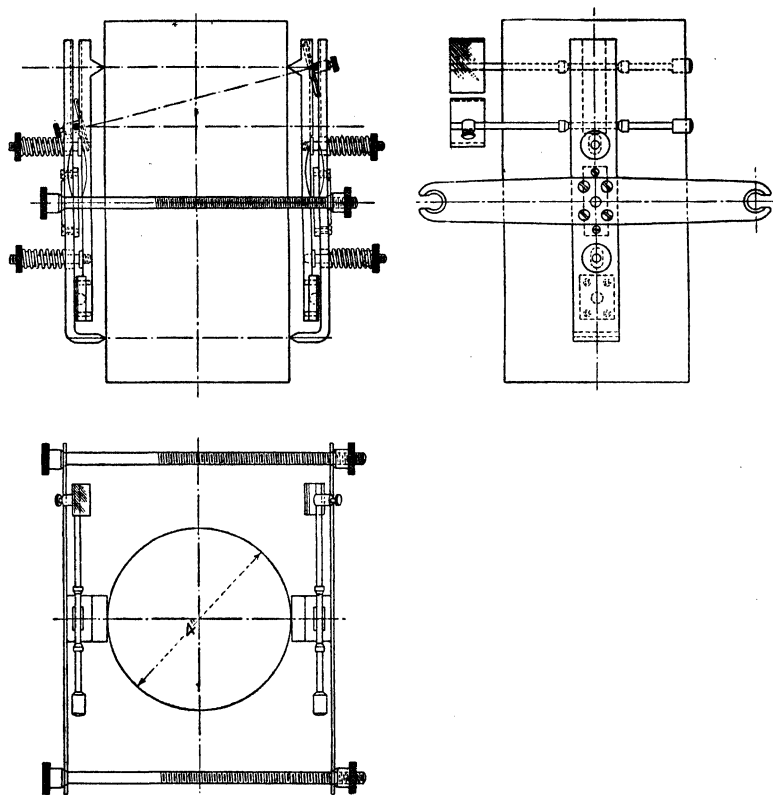



FIG. 149

be arranged for tensile or compression tests, and the principle is also applied for the measurement of lateral expansion and contraction. The instrument is shown in one form by Fig. 149, in which it is arranged for a compression test on a cement or concrete cylinder. It consists of two elements, which are

* *Engineering*, Vol. CXIII, page 684.

clamped on opposite sides of the specimen. Each element consists of two hardened and ground steel plates having a knife-edge on each; between these plates a hardened and lapped roller is inserted, and as the specimen alters in length the rollers move in opposite directions. Each roller carries a mirror, the angular movement of which is measured by a telescope and scale in the usual way. The specimen does not require to be marked off, as the knife edges are "gauge" when

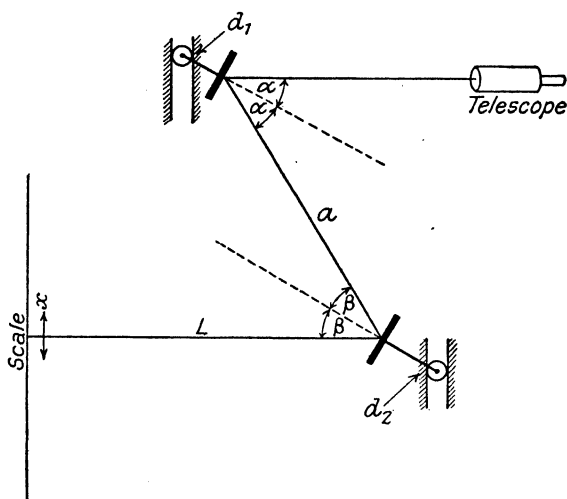



Fig. 150

the ends of the plates are flush—a gauge is supplied to check this. When using two mirrors, the scale reading indicates the mean elongation of both sides of the specimen, which automatically corrects any errors due to non-axial loading. The accuracy of the instrument depends on the diameter of the rollers, and, as measurements of these can be made to the fourth figure, the instrument is correct to that degree of accuracy.

The theory of the instrument is as follows—

Referring to Fig. 150, let d_1 and d_2 be the diameters of the rollers, α and β be the incidence and reflection angles at the mirrors, x the scale reading, a the distance between the mirrors, and L the distance from the scale to the mirror mounted on the roller of diameter d_2.

If δa and $\delta\beta$ are small opposite rotations of the mirrors, the increase in scale reading is given by

$$\delta x = 2L(\delta a + \delta\beta) + \frac{2a \,.\, \delta a \,.}{\cos\beta} \cos\beta$$

$$= 2L(\delta a + \delta\beta) + 2a \,.\, \delta a \quad . \quad . \quad (1)$$

If each extensometer element receives a deformation such that e_1 is the deformation and d_1 the corresponding roller diameter, then

$$e_1 = d_1 \,.\, \delta a \text{ and } e_2 = d_2 \,.\, \delta\beta \quad . \quad . \quad (2)$$

The deformation in the direction of the axis is given by e, where

$$e = \tfrac{1}{2}(e_1 + e_2)$$
$$= \tfrac{1}{2}(d_1 \,.\, \delta a + d_2 \,.\, \delta\beta) \quad . \quad . \quad . \quad (3)$$

If the loading is non-axial the effect will be a small proportional difference in the values of e_1 and e_2, and we have

$$e_2 = (1 + k)e_1 = (1 + k)d_1\delta a \quad . \quad . \quad (4)$$

For any small difference in the diameters of the rollers we have, by writing d in place of d_1

$$d_2 = (1 + h)d \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (2), (4), and (5)

$$(1 + h)\delta\beta = (1 + k)\delta a \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and combining (1) and (6)

$$\delta x = \left\{ 2L \,.\, \frac{2 + k + h \,.}{1 + h} + 2a \right\} \delta a \quad . \quad (7)$$

From (3), (5) and (6)

$$e = \tfrac{1}{2}d(2 + k)\delta a \quad . \quad . \quad . \quad . \quad (8)$$

Combining (7) and (8)

$$e = \frac{d\left(1 + \frac{k}{2}\right)\delta x \,.}{2L \,.\, \frac{2 + k + h \,.}{1 + h} + 2a} \quad . \quad . \quad . \quad (9)$$

If we omit the second order of small quantities, this reduces to

$$e = \frac{d\left(1 + \frac{h}{2}\right)\delta x}{4L + 2a} \quad . \quad . \quad . \quad . \quad (10)$$

or

$$e = \frac{D}{4L + 2a}\,\delta x \quad . \quad . \quad . \quad . \quad (11)$$

where D = mean roller diameter.

181. **Testing of Alloys.** Many materials, particularly the light alloys, do not exhibit a clearly defined elastic limit, limit

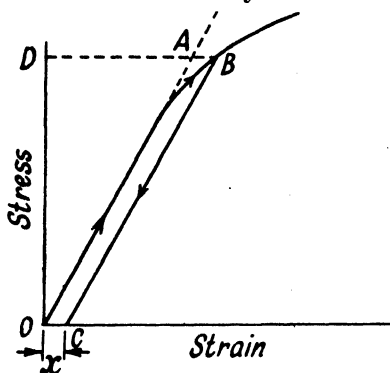



FIG. 151

of proportionality, or yield point and several methods of indicating these stresses are in use. The most widely used method is that involving a measure of permanent set and is called the "offset method."

Fig. 151 shows the stress strain relationship for a material stressed beyond the limit of proportionality and then unloaded. The slope during the unloading stage is practically similar to that during the elastic range of loading.

The offset AB is equal to the permanent set x, and thus the stress for any given amount of inelastic deformation is easily obtained from the stress-strain or load-extension diagrams. The amount x is set off on the strain or extension axis and a line CB is drawn parallel to the straight line portion of the loading curve. Through B, a line BD is drawn, parallel to OC and the stress, or load, causing the permanent set x is read off at D.

In defining a particular stress by this method, the magnitude of x is chosen from experience, and is that value which it is considered will give a stress of practical value to the designer.

The values of various offsets and the corresponding stresses, applicable to different materials, are referred to in various

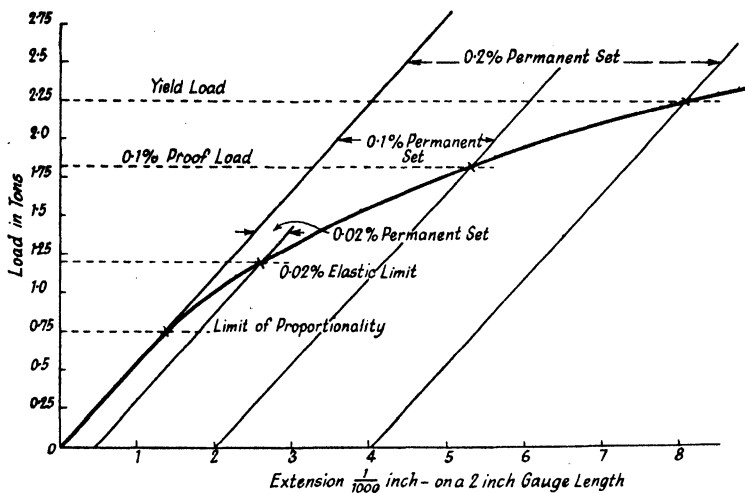



FIG. 152

British Standard Specifications, and these should be consulted for fuller details.

In the case of aluminium alloys the following offsets and stresses have been recommended, and Fig. 152 shows how they are obtained from the load-extension diagram.

The elastic limit is taken to be that stress at which there is a permanent set of 0·02 per cent; this is generally called the 0·02 per cent elastic limit. The proof stress is that stress at which there is a permanent set of 0·1 per cent and is called the 0·1 per cent proof stress. The yield point is that stress at which there is a permanent set of 0·2 per cent.

In the case of a 2 in. gauge length, the extensions corresponding to the above stresses are 0·4, 2 and 4 thousandths of an inch respectively, and thus extensometer measurements are required for the accurate production of the load-extension diagram.

The specimen from which Fig. 152 was obtained had a cross-sectional area of $\frac{1}{4}$ sq. in. From the diagram it is found that

the limit of proportionality, 0·02 per cent elastic limit, 0·1 per cent proof stress, and yield stress have the values 3·0, 4·8, 7·04, and 9·0 tons/sq. in. respectively, whilst Young's Modulus has a value of 10·24 tons/sq. in.

EXAMPLES XV

1. Make careful sketches of a "lever" testing machine, giving good scale details of shackles suitable for—
   (1) A flat plate specimen, $1\frac{1}{2}$ in. × $\frac{1}{2}$ in. section.
   (2) A round specimen 0·564 in. diameter with collars at the ends.
   (3) A round specimen 0·564 in. diameter with screwed ends.
(Lond. Univ., 1926.)

2. You are given a bar of cast iron 1·5 in. diameter and 12 in. long, and are required to find its modulus of elasticity and its breaking strength in tension. Make a dimensional sketch of the specimen you would prepare for testing. A cast iron specimen 1·000 in. diameter, when tested in tension for the first time, gave the following results—

Load (tons) . . .	0·2	0·6	1·0	1·4	1·8	2·2	2·6	3·0	2·0	1·0	0·2
Extensometer reading . .	3·62	3·95	4·29	4·67	5·04	5·46	5·85	6·31	5·41	4·53	3·79

The calibration was 5·13 scale divisions equals $\frac{1}{100}$ in. and the extensions are measured on an 8 in. length of specimen. Make a report on this test. (Lond. Univ., 1925.)

3. A tensile test on a steel tube on which was fixed a 2 in. extensometer gave the following results—

Load on specimen (tons)	0·1	0·2	0·4	0·6	0·8	1·0	1·4	1·8	2·0	2·2
Extensometer reading .	10·0	9·85	9·54	9·24	8·94	8·64	8·03	7·41	7·09	6·75
Load on specimen (tons)	2·3	2·4	2·2	2·0	1·6	1·2	0·8	0·4	0·2	
Extensometer reading .	6·5	5·75	5·90	6·19	6·78	7·36	7·94	8·57	8·89	

The tube had an external diameter of 1·125 in., and a thickness of 0·036 in. One unit of the extensometer scale = 0·0004 in. per inch length of specimen. Plot the complete load-extension diagram, and determine—
   (*a*) The initial elastic limit;
   (*b*) The modulus of elasticity.
On re-loading, what elastic limit would you expect to get? (Lond. Univ., 1916.) *Ans.*, 17·9 tons/sq. in., 13·8 × 10^3 tons/sq. in.

4. A test bar $1\frac{1}{2}$ in. diameter and gauge length 8 in. was subjected to tensile loads. A Ewing extensometer was fitted first to the 8-in. gauge points, and secondly across the bar mid-way between the gauge points. The following results were obtained—

Loads (tons) . . .	2	6	8	12	16	20
Extensometer readings (1)	4·50	5·10	5·70	6·25	6·85	7·45
,, ,, (2)	3·25	3·40	3·56	3·73	3·90	4·05

The calibrations of the extensometer gave 480, and 2,500 extensometer units per inch alteration of test-bar respectively. Find the elastic constants of the bar. (Lond. Univ., 1919.)
Ans., $E = 13{\cdot}7 \times 10^3$, $C = 5{\cdot}37 \times 10^3$, $K = 10{\cdot}15 \times 10^3$ tons/sq. in. respectively, $m = 3{\cdot}62$.

5. Describe, with sketches and index letters, the principle and use of an instrument for the measurement of the elastic strain of a metal tension test piece. Explain how you would calibrate the instrument. (A.M.I.Mech.E., 1925.)

6. Discuss the effect of increasing temperature on the yield point, ultimate strength, and elastic modulus of (*a*) steel, (*b*) brass. Sketch curves, showing the nature of the variations, and state a few numerical values to mark the different limits. (Lond. Univ., 1921.)

7. Describe an apparatus by which the maximum shear strength of a bar of solid steel may be determined. (Lond. Univ., 1919.)

8. Discuss the effect of "notching" on the tenacity of a test-bar. A bar of mild steel $8\frac{1}{2}$ in. long and $1\frac{1}{4}$ in. diameter has a groove of dimensions $\frac{1}{2}$ in. wide by $\frac{1}{8}$ in. deep at the centre of its length. Another bar of the same material has the same length and a uniform diameter of 1 in. Compare the resiliences of the two bars for the same maximum tensile stress. (Lond. Univ., 1918.)

Ans., $\frac{1}{1 \cdot 52}$

9. What is meant by "overstrained material"? What tests would you propose to perform in order to ascertain if a sample of mild steel has been overstrained? Explain how the results of the tests would enable you to decide as to the state of the steel. (Lond. Univ., 1922.)

10. A beam of pitch pine was tested by bending on a 4 ft. span, the load being applied at the middle of the span. The depth of cross-section was 2·52 in. and the width 1·81 in. The readings of load in tons and deflection in inches up to the breaking load of 1·26 tons were as follows—

Load (tons)	0·2	0·4	0·6	0·7	0·8	0·9	1·0	1·11	1·2	1·25	1·26
Deflection (in.)	0·19	0·37	0·56	0·65	0·74	0·86	0·97	1·11	1·28	1·41	1·45 broke

Find the value of the modulus of elasticity of the wood and the limiting longitudinal stress. Estimate the least height from which a load of 200 lb. could have been dropped on to the middle of this beam supported over a 4 ft. span in order to fracture the beam. (Lond. Univ., 1925.)

Ans., E = 2,310,000 lb./sq. in., 11,250 lb./sq. in., 10·5 in.

11. The following data are taken from a torsion test of a specimen of wrought iron. Diameter of specimen 0·48 in. Length under test 3·0 in.—

Twisting moment (in. lb.)	48·6	97·2	145·8	194·4	243·0	291·6	340·2
Angle of twist (degrees)	0·15	0·25	0·40	0·60	0·70	0·80	0·95
Twisting moment (in. lb.)	388·8	437·4	486·0	534·6			
Angle of twist (degrees)	1·10	1·25	1·45	2·35			

From the above results, plot a diagram of shear stress against angle of twist and determine the modulus of transverse elasticity in tons per square inch. What is the probable limit of elastic shear stress for this piece of material? (A.M.I.Mech.E., 1924.) *Ans.*, $11 \cdot 46 \times 10^6$ lb./sq. in., 20,700 lb./sq. in.

12. Sketch some form of compressometer suitable for determining the modulus of elasticity of a stone or concrete in compression. A concrete block 4 in. square is loaded in a testing machine and strains measured by means of a compressometer, and the following results obtained—

Load (lbs.)	870	2,840	5,010	6,210	8,830	9,960
Compression on 8 in. length (inches)	0·000152	0·000547	0·000985	0·00123	0.00177	0·00193

Plot the load strain and stress strain curve, and determine a value of E at a stress of 500 lb. per square inch. (Lond. Univ., 1923.)

Ans., $2{\cdot}47 \times 10^6$ lb./sq. in.

13. State the most important commercial applications of the Brinell hardness test and briefly describe, with a diagram, a satisfactory type of Brinell hardness testing machine. Calculate the Brinell hardness number of a piece of metal when a load of 3,000 kg. is used with a ball of 10 mm., and the diameter of the indentation produced on the metal is 6 mm. (A.M.I.C.E., 1926.) *Ans.*, 95·5.

14. Show by sketches how the following materials fail in tension, compression, and torsion—

(1) A 3 per cent nickel alloy steel quenched in oil and tempered at 650° C.
(2) Cast iron.

Discuss in detail the failure of the cast iron specimens and indicate the type of stress that has caused failure. Hence deduce from the types of failure the relative strengths of cast iron in (*a*) tension, (*b*) torsion, (*c*) compression. (Lond. Univ., 1925.)

15. Briefly describe a machine whereby a "shock" or impact test may be made upon a material. Sketch a specimen prepared for such a test and say how the result of the test would enable you to judge of the quality of the material. (Lond. Univ., 1923.)

16. Give a brief explanation of the chief results which have been obtained from recent research work on the effect of repeated and alternating stresses. How are these results employed when determining the working stresses to adopt in designing machines or structures in which repeated or alternating stresses occur. (Lond. Univ., 1916.)

17. Describe with the help of sketches any method of testing the strength of a given sample of steel when subjected to repetitions of stress, and state fully what you know of *one* of the following—

(*a*) Gerber's parabolic relation, *or*
(*b*) Strain method of determining fatigue ranges, *or*
(*c*) Effect of rapid changes of section and surface conditions on the limiting ranges of stress. (A.M.I.C.E., 1926.)

18. Explain fully the procedure adopted in preparing cement briquettes for tension testing. Make a neat diagrammatic sketch of any cement-testing machine with which you are acquainted, and describe how a test is carried out. (A M.I.Mech.E., 1918.)

19. The following observations were made during a compression test of a small mild steel cylinder: Diameter of cylinder 0·75 in., length of cylinder 1·50 in., load applied axially.

Load (tons) .	5	10	10·6	15	20	30	40	50
Compression (inches) .	0·006	0·012	0·050	0·122	0·237	0·492	0·701	0·836

Assuming that the volume of the material remains constant, plot curves on a compression base, showing the variation of load and of mean stress in the material. What deductions would you make from these curves? (Lond. Univ., 1923.)

20. In making a determination of ultimate strength, why is it so much more important to ensure that the pull shall be axial when testing cast iron than when testing mild steel?

A rod of cast iron 1 in. diameter was subjected simultaneously to an axial tensile pull and a twist. The iron fractured along a helix which was inclined

62° to the axis of the rod. If the tenacity of cast iron is 8·5 tons per square inch, find the probable magnitudes of the axial pull and the torque. (Lond. Univ., 1922.) *Ans.*, 4·78 tons, and 0·885 ton ins.

21. The following data were obtained during a test on an aluminium alloy bar of $\frac{1}{4}$ in. cross-sectional area and gauge length 2 in.

Load (tons)	0·25	0·5	0·75	1·0	1·25	1·5	1·75	2·0	2·25
Extension ($\frac{1}{1000}$ in.)	0·44	0·89	1·33	1·78	2·23	2·68	3·3	4·05	5·05

Load (tons)	2·5	2·75	3·0	3·25	Extension at fracture
Extension ($\frac{1}{1000}$ in.)	6·25	7·6	9·3	12·0	0·36 in.

Calculate the value of Young's Modulus, the limit of proportionality stress, the 0·02 per cent elastic limit stress, the 0·1 per cent proof stress, the 0·2 per cent yield stress, and the percentage elongation for the alloy.

Ans., 10·03 × 10^6, 6·0, 7·8, 10·48, and 12 tons/sq. in. respectively. 18 per cent.

22. The following observations were made during combined bending and torsion tests on seven bars of similar material and of similar dimensions. Each bar was submitted to a constant uniform bending moment and the twisting moment increased until the limit of proportionality was reached.

Bar number	1	2	3	4	5	6	7
M lb./in.	100	200	300	400	500	600	700
T lb./in.	850	830	800	750	700	610	490

State which theory of elastic failure seems to govern this form of loading.

Ans., Guest's theory. Mean value of T_e = 854·5 lb./in.

LINEAR COEFFICIENTS OF EXPANSION

Aluminium (cast)	0·0000203 per °C.
Brass	0·0000185 ,,
Copper	0·0000185 ,,
Glass	0·0000077 ,,
Iron (cast)	0·0000108 ,,
Iron (wrought)	0·0000131 ,,
Lead	0·0000290 ,,
Masonry and cement	0·0000138 ,,
Nickel	0·0000120 ,,
Steel	0·0000120 ,,
Tin	0·0000210 ,,
Zinc	0·0000290 ,,

FACTORS OF SAFETY

	Dead Load	Live Load	Shock
Cast iron	4	6–10	10–15
Wrought iron	3	5– 8	9–13
Steel (mild)	3	5– 8	9–13
Copper	5	6– 9	10–15
Timber	7	9–14	14–20

CHAPTER XVI

CREEP OF METALS

182. DURING recent years there has been a rapid rise in the working temperatures of steam boilers and turbines, in certain chemical processes, and in other directions. The behaviour of metals at such temperatures has occupied the attention of many investigators, and although the results of such investigations have not been in complete agreement, certain facts have been revealed which are of the utmost importance to designers

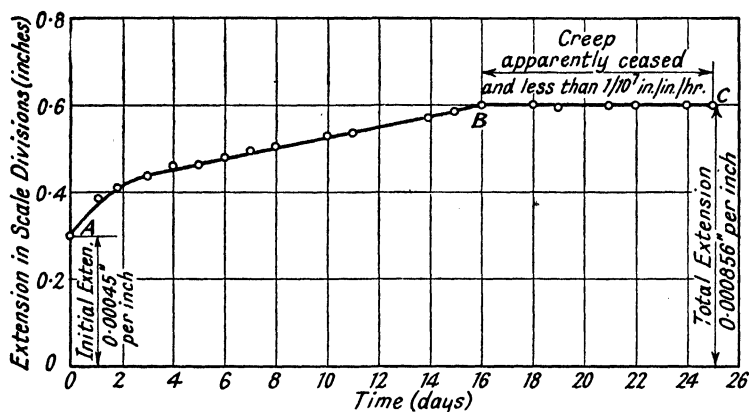



FIG. 153

of plants which are required to work at continuous high temperatures.

At high temperatures it is found that the ordinary condition of elasticity of metals changes to a state of viscous flow whereby continuous deformation or *creep* may proceed at small rates.

This characteristic is illustrated in Fig. 153. A steel specimen heated to 850° F. was stressed up to 5 tons/sq. in., and the initial extension OA noted. The stress and temperature were kept constant and the extension measured at intervals of 48 hours. The curve ABC represents the total extension plotted to a time base. It will be noted that the rate of creep is more rapid for the first period of 8 days than for succeeding periods,

the maximum value being *practically* reached after 16 days, the growth after this period being very slow indeed.

At higher temperatures smaller stresses will still produce creep, and thus it is not possible to define a breaking strength for metals at high temperatures. Some investigators state that at a temperature above 1000° F. any stress, however

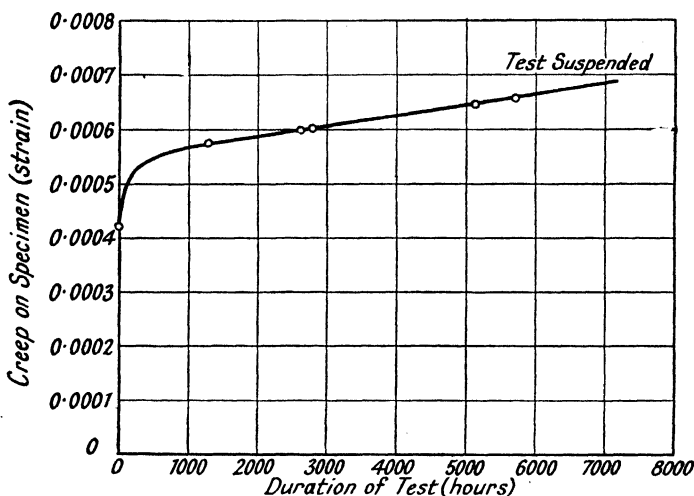



FIG. 154

small, would cause continuous flow or creep in such materials as carbon steels.

In order to obtain information on creep characteristics, many forms of creep tests have been devised. The installations show considerable variation in design as they are often required to suit particular conditions, but a clear distinction can be drawn between short-time and long-time testing methods. The duration involved in the latter method is an obvious drawback, but short-time tests suffer in general from the disadvantage that they may not give more than an indication of the relative order of creep of a range of materials. Thus the general opinion is that long-time creep tests are necessary for obtaining data which can be used in service.

183. **Methods of Creep Measurement.** In measuring creep it is usual to state the stress in tons/sq. in. at a given temperature which will cause a deformation of a certain number of inches

per inch (strain) in a given time, or a deformation of a given percentage in a given time.

The former method is illustrated by Fig. 154.

The graph is taken from a paper by Bailey and Roberts* and represents a test on a nickel-chrome-molybdenum steel specimen carrying a stress of 5 tons/sq. in. at a continuous temperature of 450° C. It is of interest to note in this test that stoppages took place at certain intervals and these stoppages necessitated the re-setting of the strain measuring instruments; it will be observed from the graph that the rate of creep continued the same after as before the stoppage.

The second method of creep measurement is illustrated in the accompanying table taken from a paper on Copper Alloys in Engineering.†

CREEP PROPERTIES OF SOME COPPER ALLOYS

Material	Temp. °C.	Stress in tons/sq. in. to produce a creep rate of		Reference
		0·01 per cent per 1,000 hours	0·1 per cent per 1,000 hours	
60 Cu, 40 Zn (rolled)	150 205	4·0 0·9	5·4 2·1	Clark and White
Naval Brass: 59 Cu 40 Zn, 1 Sn (rolled)	150 205	5·4 1·6	6·7 2·5	,,
70 Cu, 30 Zn (rolled)	205 315	5·7 0·1	8·0 0·4	,,
85 Cu, 15 Zn (rolled)	205 315	3·9 0·45	5·4 1·2	,,
Gunmetal: 87 Cu, 11 Sn 2 Zn (Cast Bar)	205 260 315	5·0 1·8 0·8	7·0 3·0 1·4	Stewart
Gunmetal: 88 Cu 10 Sn, 2 Zn (specimen cut from large casting)	205 260 315	5·0 2·7 1·3	7·0 4·0 2·0	,,
87 Cu, 7 Sn, 5 Zn 1 Pb. (cast test bar)	205 260 315	5·0 2·3 1·1	7·0 4·0 2·2	,,
90 Cu, 6 Sn, 2 Zn 2 Pb, (specimen cut from large casting)	205 260 315	4·0 2·7 1·4	6·0 4·0 1·8	,,
92·5 Cu, 7·5 Al (die-cast)	290 450	— —	4·4 1·3	Kanter

* *Proc. Inst. Mech. Eng.*, 1932.

† Copper Development Association Publication No. 32.

The figures also serve to show the relative creep values of rolled and cast alloys.

184. **Comparison of Tensile and Creep Tests.** The figures shown in the following table are also taken from the previously mentioned paper by Bailey and Roberts, and the tests were carried out at a temperature of 500° C. The method of carrying out the tests was as follows—

Tensile Tests. The stresses recorded were the maximum developed when the specimens were tested with a straining rate of 0·001 inch per inch per minute.

Creep Tests. The stresses recorded were the values giving a minimum creep rate of 10^{-5} strain per hour.

Type of Steel		Max. Stress Tensile Test Tons/sq. in.		Creep Stress for Min. Strain Rate of 10^{-5} per Hour Tons/sq. in.		Remarks
		As Received	After 2,000 Hours at 560°C.	As Received	After 2,000 Hours	
"Armco" iron		7·55	—	2·85	—	
0·15 C (annealed)		13·56	—	6·25	—	
0·15 C (normalized)		13·45	12·23	6·25	6·0	
0·25 C (forged)		14·76	—	5·15	—	
0·3 C (cast)		15·05	13·8	5·0	4·1	
0·4 C (forged)		22·9	17·65	7·7	5·9	
5 per cent nickel	(1)	17·15	13·9	2·75	2·7	Cold drawn
	(2)	17·9	—	2·7	—	
14 per cent Cr Stainless	(1)	25·2	22·4	8·25	8·25	
	(2)	24·04	—	8·25	—	
Ni, Cr, Mo.	(1)	38·5	27·7	15·0	9·0	
	(2)	41·0	—	16·5	—	
35 Ni, 10 Cr		38·1	32·2	17·25	17·0	

From the above it would appear that the tensile test serves as a rough guide in sorting materials and that the maximum stress is a measure of the creep resistance of the material.

185. **Short Creep Tests.** The time required for the performance of creep tests increases so rapidly with reduction in the minimum creep rate for which a stress-temperature relation may be desired that the requisite test duration may run into months or longer. Hence if experimental stress data for a minimum creep rate are desired, either results for a relatively large creep rate will have to be accepted or else means will have to be determined to shorten the period occupied in reaching the minimum creep rate. With the object of saving time, a creep

rate for short tests and for comparison purposes should be chosen large enough to ensure satisfactory results. A minimum creep rate of 10^{-4} or 10^{-5} strain per hour will be found to meet the need for economy in time and at the same time give a figure for the resistance of the material suitable for comparative purposes.

186. **Effect of Carbon on Creep Properties of Steels.** It is generally accepted that increasing the carbon content has no effect on the creep properties of steels. This statement is admirably supported by the figures given in the following table which has been taken from a report by the British and Allied Industries Research Association.*

The tests were carried out on wrought carbon steels at a temperature of 450° C. under a stress of 8 tons/sq. in. The first four steels were tested at a creep rate of $0{\cdot}8 \times 10^{-6}$ strain per hour. The 0·4 per cent carbon steels were tested at a creep rate of $1{\cdot}5 \times 10^{-6}$ strain per hour, and for the 0·61 per cent carbon steel the creep rate was $12{\cdot}0 \times 10^{-6}$ strain per hour.

Steel No.	Carbon Content per cm.	Heat Treatment before Test	Ultimate Stress at Air Temperature Tons/sq.in.	Creep Properties			
				Total Creep at 20 Days, per cent	Average Total Creep at 20 Days, per cent	Rate of Creep at End of 20 Days, Inches per inch per hour $\times 10^6$	Average Rate of Creep at End of 20 Days, Inches per inch per hour $\times 10^6$
1	0·13	Air cooled from 950°C.	27·2	0·36	0·19	1·3	0·8
21	0·15		30·9	0·12		0·8	
19	0·16		29·0	0·13		0·8	
20	0·20		31·8	0·14		0·3	
4	0·37	Air cooled from 900°C.	37·8	0·13	0·27	0·4	1·5
14	0·40		39·0	0·20		1·7	
9	0·42		38·3	0·38		2·1	
10	0·40		39·4	0·22		1·5	
16	0·44		40·3	0·29		1·5	
22	0·40		40·0	0·40		2·1	
3	0·61	Air cooled from 875°C.	48·6	0·84		12·0	

187. **Effect of Molybdenum on Creep Properties of Steels.** The addition of molybdenum to carbon steel effects a considerable improvement in the creep resistance and the most

* *Proc. Inst. Mech. Eng.*, 1939.

advantageous amount is usually about 0·5 per cent. From tests carried out on several air-cooled steels by the British and Allied Industries Research Association it would appear that a slight improvement is obtained by increasing the molybdenum up to about 1 per cent.

188. **Effect of Molybdenum and an Additional Element on Creep Properties of Steel.** Although the addition of molydenum to carbon steels has the effect of increasing the creep resistance, microscopic examination has shown that test specimens fractured because of the development of intercrystalline cracks, and that the greater the creep resistance the greater the tendency to cracking. The previously mentioned research committee investigated the effect of the addition of various elements to molybdenum steel and the results are given in the following table.

Steel No.	Carbon Content, per cent	Molybdenum Content, per cent	Additional Alloying Element, per cent	Heat Treatment before Test	Ultimate Stress at Air Temperature Tons/sq. in.	Total Creep at 40 Days, per cent	Rate of Creep at 40 Days, Inches per inch per hour $\times 10^6$
23	0·18	0·51	—	Air cooled from 975°C.	31·2	0·195	0·52
30	0·14	0·82	—	Air cooled from 975°C.	33·0	—	$<$ 0·3
24	0·14	0·54	Vanadium 0·20	Air cooled from 1000°C.	37·8	0·098	$<$ 0·3
25	0·19	0·51	Tungsten 0·57	Air cooled from 925°C.	35·7	0·135	0·73
26	0·16	0·50	Copper 0·25	Air cooled from 975°C.	32·1	0·30	2·1
27	0·198	0·48	Manganese 1·48	Air cooled from 950°C.	46·8	0·43	2·7
28	0·13	0·52	Titanium 0·24	Air cooled from 1000°C.	30·2	2·57	11·0

These tests were carried out at a temperature of 550° C. and under a stress of 9 tons/sq. in. It would appear from the figures that no advantage in creep resistance is gained by the addition of the above percentages of tungsten, copper, manganese, or titanium. The addition of 0·2 per cent vanadium appears to be advantageous.

189. **Heat Treatment—Spheroidization.** A carbon steel with a carbon content up to 0·9 per cent in the fully annealed and normalized conditions consists of grains of ferrite and pearlite. The pearlite has a lamellar structure arising from alternating bands of ferrite and iron carbide or cementite. If the steel is heated for a sufficient time at a temperature below the lower

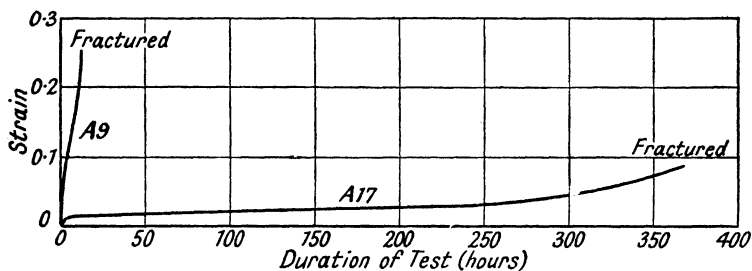



Fig. 155

critical temperature (700° C. in carbon steels) the lamellar structure of the pearlite is changed, the iron carbide lamellae breaking up into collections of small masses. The general form of the masses tends to be globular and the change from the lamellar structure is known as spheroidization of the carbide or the carbide is said to be *spheroidized.* As this change is accompanied by a substantial reduction in the resistance to creep its importance will be appreciated. The effect of spheroidization on the creep resistance of alloy steels is also pronounced as shown by Fig. 155.

The diagram shows creep test curves on a 0·5 per cent molybdenum steel tested under similar conditions by Bailey and Roberts.* In the case of A17 the structure is lamellar resulting from heating at 850° C. for 6 hours and in the case of A9 heating for 6 hours at 650° C. resulted in the spheroidized condition. The minimum creep rate for A9 is 240 times that of A17, the rates being $1{\cdot}79 \times 10^{-2}$ per hour and $7{\cdot}54 \times 10^{-5}$ per hour

* *Proc. Inst. Mech. Eng.*, 1932.

respectively. The tests were carried out at 550° C. under a stress of 14 tons/sq. in., and in each case air-cooling was adopted.

The effect of heat treatment on the creep properties of the carbon, carbon-molybdenum, and on the complex molybdenum steels already mentioned, was studied and the following facts emerged.

Carbon steels which have a coarse-grained structure gave results at 451° C., which did not greatly differ from those for the normalized specimen, but steels which were spheroidized by heating for 18 hours at 625° C. gave greatly diminished results.

Carbon-molybdenum steels which had been oil quenched and tempered for 6 hours at 625° C. when tested under a stress of 9 tons/sq. in. and at 550° C. showed slight improvement on the air-cooled condition when the carbon content was low. Considerable deterioration took place with 0·4 per cent carbon steels. Spheroidization which was produced by heating air-cooled material for 14 days at 625° C. showed a marked decrease in creep resistance.

Complex molybdenum steels which had been air-cooled showed the best creep properties, but oil quenching and heat treatment gave nearly as good results for the plain molybdenum steel and for the vanadium-molybdenum steel. Slow cooling in a furnace at 100° C. per hour gave a decrease in creep resistance, and spheroidization, produced by heating for 14 days at 625° C., showed a marked decrease except for the vanadium-molybdenum steel which was not very susceptible to any of the above treatments.

CHAPTER XVII

PHOTO-ELASTIC METHOD OF STRESS ANALYSIS

190. THE probable maximum stress in a component as found by mathematical methods is often far from accurate, since the intricate shape of the component may render the use of such methods inapplicable. Higher values of stress than those

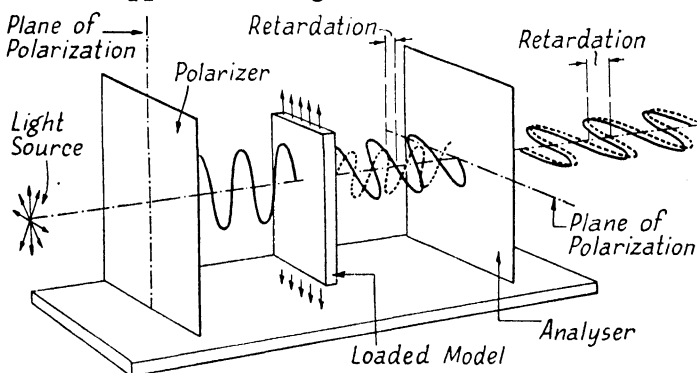



FIG. 156

calculated by such methods are often concentrated at unsuspected points and these are clearly revealed by photo-elasticity.

The photo-elastic method of stress analysis not only enables the designer to obtain a rapid qualitative picture of stress distribution in a loaded component, but also provides quantitative information from which the stresses may be evaluated.

191. **Polarized Light.** A ray of light is assumed to travel in wave formation in any number of directions transverse to the direction of the beam. If the vibrations of the ray are confined to one plane the resultant light is then referred to as plane-polarized light.

When a plane-polarized light ray is passed into a loaded model of suitable plastic material the ray is split into two component rays vibrating in the directions of the principal stresses at the point. This is known as double refraction, and the velocities of the component rays are unequal, being dependent on the magnitude of the principal stresses.

The component rays on emergence from the model may be recombined in one plane by passing them through a second polarizing element called an analyser. The plane of polarization of the analyser is at right angles to that of the former polarizing element. Interference will occur if the rays are out of phase, and if they are projected on a screen the interference is revealed as a black spot. The behaviour of the various rays is shown in Fig. 156.

When plane-polarized white light is used and the source

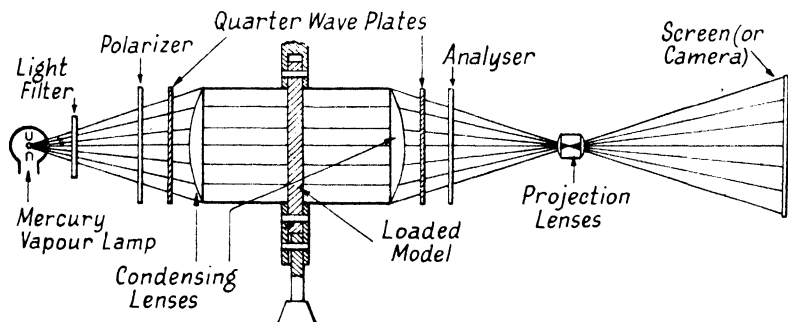



Fig. 157

illuminates the entire model, then the image of the model, projected on the screen, will be covered by a series of brightly coloured bands. These are known as isochromatics. If, however, the light is monochromatic the brightly coloured bands are replaced by black lines called "fringes." The monochromatic source of light is preferred to the white light source as the alternate black and monochromatic bands are easily distinguishable (Fig. 159).

If the plane of the incident polarized light ray is made to coincide with the direction of one of the principal stresses the light will pass straight through the model and will be annihilated by the analyser, resulting in a black spot on the screen. This occurs at every point at which the principal stresses in the model have the same directions, and the loci of these points is a black line on the image called an isoclinic. Isoclinics are useful in determining the directions of the principal stresses. Fig. 156 shows a diagrammatic representation of the apparatus described, and a series of isochromatics for a pair of wheel teeth are shown in Fig. 159. The figure shows clearly the stress-concentrations at the point of contact of the

teeth and also at the fillet on the compression stressed side of the teeth.

If a quarter-wave plate is inserted, on each side, between the model and the polarizing element in the apparatus shown in Fig. 156, then isochromatic lines only will be shown. The arrangement for such lines is shown in Fig. 157. The quarter-wave plates can readily be removed if isoclinics are required. Isoclinic lines on the image of a loaded beam are shown in Fig. 160.

192. **Retardation of Component Rays.** The amplitude of a ray of light after a time t may be expressed by the equation—

$$a = A \sin wt \qquad (1)$$

when the amplitude is zero at zero time. If this condition does not hold the equation is of the form $a = A \sin w(t - t_0)$ where t_0 is the time at which the amplitude becomes zero. The polarizing ray on entering the loaded model is broken up into component rays in the directions of the principal stresses, as shown in Fig. 158. These component rays may be represented by OB and OC where the vector a is the amplitude of the entering polarized ray. The values of OB and OC are given by $a \cos \alpha$ and $a \sin \alpha$ respectively, or substituting in equation (1)—

Fig. 158

$$OB = A \cos \alpha \sin wt \qquad (2)$$

$$OC = A \sin \alpha \sin wt \qquad (3)$$

If V_1 and V_2 are the respective velocities of the component rays through the model whose thickness is x, and t_1 and t_2 are the respective times for the rays to travel through the model then—

$$t_1 = \frac{x}{V_1} \qquad (4)$$

$$t_2 = \frac{x}{V_2} \qquad (5)$$

FIG. 159

FIG. 160

and the equations of the displacements on emergence are given by—

$$d_1 = A \cos \alpha \sin w(t - t_1) \quad . \quad . \quad . \quad . \quad (6)$$
$$d_2 = A \sin \alpha \sin w(t - t_2) \quad . \quad . \quad . \quad . \quad (7)$$

The time lag or retardation of one component ray relative to the other is—

$$\begin{aligned} t_1 - t_2 &= \frac{x}{V_1} - \frac{x}{V_2} \\ &= \frac{V_2 - V_1}{V_1 V_2} x \\ &= \frac{(V_2 - V) - (V_1 - V)}{V_1 V_2} x \quad . \quad . \quad . \quad . \quad (8) \end{aligned}$$

Since $V_1 V_2$ is practically constant and the change in velocity is proportional to the principal stress, then the retardation is approximately proportional to the difference between the two principal stresses, i.e.

$$t_1 - t_2 = C\,(f_{n_2} - f_{n_1}) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

After passing through the analyser the component rays *OB* and *OC* will be resolved into two components, as shown in Fig. 161. The components *OG* and *OF* will be annihilated and the component rays *OE* and *OD* will be transmitted. The amplitudes of these emerging rays are—

FIG. 161

$$\begin{aligned} OE &= OB \sin \alpha \\ &= a \cos \alpha \sin \alpha \\ &= \frac{a}{2} \sin 2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10) \end{aligned}$$

$$\begin{aligned} OD &= OC \cos \alpha \\ &= a \sin \alpha \cos \alpha \\ &= \frac{a}{2} \sin 2\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

Thus the rays OE and OD have the same amplitude, and their equations are given by—

$$d_1' = \frac{a}{2} \sin 2\alpha \sin w(t - t_1) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$d_2' = \frac{a}{2} \sin 2\alpha \sin w(t - t_2) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If these rays are out of phase the resultant ray will be the vector difference of the component rays, and its equation is—

$$d'' = \frac{a}{2} \sin 2\alpha \,.\, [\sin w(t - t_1) - \sin w(t - t_2)] \quad . \quad . \quad (14)$$

193. **Conditions for Formation of Isoclinics and Fringes.** From equation (14) it is seen that the value of d'' will be zero, i.e. no light will be transmitted, for two conditions, as follows—

(1) For a value of $\alpha = 0$ which corresponds to the condition that the plane of polarization corresponds with the direction of one of the principal stresses. One component ray emerges from the model and is eliminated by the analyser.

(2) When $\sin w(t - t_1) = \sin w(t - t_2)$. This condition is satisfied when,

$$w(t - t_1) = w(t - t_2) + 2n\pi \quad . \quad . \quad . \quad . \quad (15)$$

where n is an integer. It follows then that

$$w(t_2 - t_1) = 2n\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Substituting the value of $(t_1 - t_2)$ from (9)

$$C\,(f_{n_1} - f_{n_2}) = 2n\pi$$

or

$$f_{n_1} - f_{n_2} = \frac{2n\pi}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and from page 21 (5) the value of the maximum shear stress at the point is given by—

$$f_{s_{max}} = \frac{f_{n_1} - f_{n_2}}{2}$$

$$\therefore \qquad f_{s_{max}} = \frac{n\pi}{C}$$

or from (8)

$$f_{s_{max}} = \frac{nK}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

K is a constant for the material and is called the fringe value.

194. **Evaluation of Principal Stresses.** The first condition gives a point on an isoclinic and enables the directions of the principal stresses at the point to be found. The polarizer and the analyser are rotated together until a black spot appears on the image at the point. The directions of the principal stresses correspond to the axes of polarization of the polarizer and analyser.

From the second condition no light is transmitted when equation (16) holds, and hence when equation (18) holds or

$$f_{n_1} - f_{n_2} = \frac{2nK}{x} \qquad (19)$$

Thus for every integral value of n a series of fringes will be formed. The first fringe formed as load is applied, for a value of n equal to unity, is called the first order fringe, and for a value of n equal to 2 the fringe is called the second order fringe, etc.

Several methods are employed for determining the actual value of the principal stresses at a point. Two methods are given, and for each the value of K is necessary.

A strip of material similar to that forming the model and of the same thickness as the model is loaded in such a manner that the maximum shearing stress at a point can be calculated directly. Its fringe value is noted and from equation (18) the value of K is found. At any desired point in the loaded model the difference between the principal stresses is found by means of equation (19).

An extensometer, usually of the optical type, is used to obtain the strain in the model at the point in a direction perpendicular to the surface of the model.

$$e_3 = \frac{f_{n_1}}{mE} + \frac{f_{n_2}}{mE} \qquad (20)$$

from which

$$f_{n_1} + f_{n_2} = mEe_3 \qquad (21)$$

m and E can be found for the material of the model, and equation (21) combined with equation (19) enables the principal stresses to be calculated.

The second method employs successive integration and is derived from a knowledge of the shear stress difference on a small block of material.

Fig. 162 shows the variation of stress on such a block taken

from a loaded component of uniform thickness t. Resolving the forces in a vertical direction, then—

$$(f_y + df_y)tdx - f_y\,tdx + S_y tdy - (S_y + dS_y)tdy = 0$$

or
$$df_y \,.\, dx - dS_y dy = 0$$

and
$$df_y = \frac{dS_y}{dx} \,.\, dy \qquad (22)$$

hence
$$f_{y_0} - f_{y_1} = \int_{y_1}^{y_0} \frac{dS_y}{dx} \,.\, dy \qquad (23)$$

Thus the difference between the normal stresses on two parallel planes is equal to the rate of change of shearing stress at right

Fig. 162

angles to the planes multiplied by the distance between the planes.

To evaluate the stresses in equation (23), a straight line is drawn across the component through the point at which the stress is desired and the direction of the line is taken as the y direction. The line is next divided into a number of segments each of length $\triangle y$. The shear stress S_y at each end of the segments is found from page 21, equations (4) and (5), which give—

$$S_y = \tfrac{1}{2}(f_{n_1} - f_{n_2}) \sin 2\,\theta$$
$$= S_{y_{max}} \sin 2\,\theta \qquad (24)$$

θ is found from the isoclinic, and $S_{y_{max}}$ is evaluated by means of equation (18).

The value of $\frac{dS_y}{dx}$ can be determined at the end of each segment, and starting at a boundary, where f_{y_0} is zero, successive values of f_{y_1} can be evaluated by means of equation (23).

A line is next drawn through the point in a perpendicular or x direction and following a similar procedure the following equation holds—

$$f_{x_0} - f_{x_1} = \int_{x_1}^{x_0} \frac{dSx}{dy} \, . \, dx \qquad . \quad . \quad . \quad (25)$$

and hence successive values of f_{x_1} can be found.

Thus a complete picture of the stress situation at any point in a component may be obtained and hence the principal stresses can be evaluated.

ELASTIC CONSTANTS

Material	Ultimate Strength, lb./sq. in.			$E \div 10^6$ lb./sq. in.	$C \div 10^6$ lb./sq. in.	Density, lb./cub. in.
	Tension	Compression	Shear			
Cast iron	10,000–30,000	60,000–140,000	16,000	14–23	8–11	0·265
Wrought iron	36,000–67,000	40,000	40,000	25–29	11–14	0·281
Mild steel	60,000–90,000	65,000	50,000	29–31	12–14	0·284
Duralumin	60,000	50,000	36,000	10	3·8	0·102
Aluminium	18,000	—	13,000	12	—	0·093
Copper (cast)	25,000	36,000	26,000	10	—	0·311
Copper (rolled)	36,000	47,000	30,000	—	5	0·322
Copper (wire)	55,000	—	—	17	—	0·321
Gunmetal	32,000	—	—	—	—	0·309
Phosphor bronze	40,000	—	—	—	—	—
Manganese bronze	60,000	—	—	—	—	—
Brass	25,000	10,500	16,000	9–14	5·5	0·306
Ash	13,000	7,000	1,700	1·5	0·15	46 lb. per cub. ft.
Beech	12,000	6,000	1,700	1·4	—	44 lb. ,,
Elm	13,000	5,500	1,600	1·0	—	34 lb. ,,
Pine	9,000	5,000	800	1·5	—	35 lb. ,,
Mahogany	11,000	6,000	1,350	1·3	0·1	45 lb. ,,
Lignum-vitae	11,000	10,000	—	1·0	—	80 lb. ,,
Oak	13,000	7,000	1,800	1·5	—	54 lb. ,,
Teak	8,000	—	—	2·35	—	45 lb. ,,

TYPES OF STEELS—THEIR PROPERTIES AND USES*

Type of Steel	Essential Properties	Essential Analysis	General Uses
1. *Structural steels*—			
Ordinary structural	28–32 tons/sq. in. tensile	0·24 per cent carbon, no other alloys	Material normally employed for shipbuilding, bridge building, etc.
High-tensile structural	Up to 40 tons/sq. in. tensile	Silicon up to $1\frac{1}{2}$ per cent	Several bridges erected in this steel between 1910 and 1915
High-tensile structural	Up to 45 tons/sq. in. tensile	Manganese up to $1\frac{1}{2}$ per cent	More reliable than silicon steel
High-tensile structural, with improved corrosion-resisting properties	38–42 tons/sq. in. tensile	$\frac{1}{4}$–$\frac{1}{2}$ per cent copper, $\frac{3}{4}$–1 per cent chromium Also various other combinations of manganese up to $1\frac{1}{2}$ per cent, chromium up to 1 per cent, and copper up to $\frac{1}{2}$ per cent	The most recent development in structural materials
2. *Carbon steels*—			
For engineering purposes	24–70 tons/sq. in. tensile	From 0·12 to 0·70 per cent carbon	Used for railway locomotives, Diesel engines, gas engines, steam reciprocating engines, steam and exhaust gas turbines, electrical alternators, and general engineering forgings and castings; also for steam drums, chemical reaction chambers and die blocks

* Reproduced by courtesy of the Institution of Mechanical Engineers from a paper, "The Choice of Steels for Particular Purposes," by G. E. Wolstenholme, M.I.Mech.E. (*Proceedings of the Inst. Mech. Eng.*)

Type of Steel	Essential Properties	Essential Analysis	General Uses
3. *Low-alloy steels*—			
Manganese	40-50 tons/sq. in. tensile	0·3 per cent carbon and 1½ per cent manganese	These small alloy additions raise tensile strength and improve impact strength without appreciably affecting the ductility. Molybdenum steels also possess improved strength at temperatures up to 450° C.
Molybdenum	40–50 tons/sq. in. tensile	0·3 per cent carbon and 1½ per cent molybdenum	
Manganese-molybdenum	45–55 tons/sq. in. tensile	0·3 per cent carbon, 1½ per cent manganese, and ½ per cent molybdenum	
Manganese-nickel-molybdenum	45–55 tons/sq. in. tensile	0·3 per cent carbon, 1½ per cent manganese, ½ per cent nickel, and ½ per cent molybdenum	
4. *Medium-tensile alloy steels*—			
3 per cent nickel	45–55 tons/sq. in. tensile	0·3 per cent carbon and 3 per cent nickel	These medium-tensile steels are more reliable than the low-alloy steels. The presence of nickel improves impact strength and decreases mass effect
5 per cent nickel	45–55 tons/sq. in. tensile	0·25 per cent carbon and 5 per cent nickel	
3 per cent nickel-chromium	55–65 tons/sq. in. tensile	0·3 per cent carbon, 3 per cent nickel, and ¾ per cent chromium	
Chromium-vanadium	55–65 tons/sq. in. tensile	0·3 per cent carbon, 1½ per cent chromium, and ¼ per cent vanadium	
5. *High-tensile alloy steels*—			
3 per cent nickel-chromium-molybdenum	65–75 tons/sq. in. tensile	¾ per cent chromium and ½ per cent molybdenum	The most reliable of all steels for medium high-tensile parts
1½ per cent nickel-chromium	100tons/sq. in. tensile	0·4 per cent carbon, 1½ per cent nickel, and 1 per cent chromium	Reaches 100 tons/sq. in. tensile strength, but only at expense of reliability

Type of Steel	Essential Properties	Essential Analysis	General Uses
5. *High-tensile alloy steels (contd.)*—			
4½ per cent nickel-chromium	100 tons/sq. in. tensile	0·3 per cent carbon, 4½ per cent nickel, and 1¼ per cent chromium	The most reliable steel of 100 tons/sq. in. tensile strength for gears and axles
6. *Case-hardening steels*—			
Carbon	30 tons/sq. in. tensile	0·1 per cent carbon	For ordinary low-tensile parts
3 per cent nickel	45 tons/sq. in. tensile	0·1 per cent carbon and 3 per cent nickel	Used extensively in motor car manufacture
5 per cent nickel	50 tons/sq. in. tensile	0·1 per cent carbon and 5 per cent nickel	A most reliable steel much used in aircraft
3 per cent nickel-chromium	55 tons/sq. in. tensile	0·1 per cent carbon, 3 per cent nickel, and ¾ per cent chromium	A tough-cored steel used in motor car and general engineering
4½ per cent nickel-chromium	80 tons/sq. in. tensile	0·1 per cent carbon, 4½ per cent nickel, and 1¼ per cent chromium	The toughest-cored steel of all, used in the manufacture of motor car engines and transmission and aero-engine construction
7. *Nitriding steels*—			
Chromium-aluminium	35–90 tons/sq. in. tensile; surface hardness of case, 1,050–1,150 Diamond hardness No.	0·2–0·5 per cent carbon, 1 per cent aluminium, and 1½ per cent chromium	For die blocks, brick press liner plates, concrete slabbing press plates, spindles, plungers, machine tool parts, motor car crankshafts, brake drums, etc.
Aluminium-free	40–70 tons/sq. in. tensile; surface hardness of case, 650–800 Diamond hardness No.	0·2–0·4 per cent carbon, 1–3 per cent chromium, and up to 1¼ per cent molybdenum	Developed essentially for aero-engine crankshafts and sleeves. Will probably soon be adopted by automobile industry

Type of Steel	Essential Properties	Essential Analysis	General Uses
8. *Spring steels*—			
Carbon	90–120 tons/sq. in. tensile	0·5–0·8 per cent carbon	For railway suspension and buffer springs and motor car suspension springs
Silico-manganese		0·5 per cent carbon, 2 per cent silicon, and 1 per cent manganese	
Chromium-vanadium		0·5 per cent carbon, 1½ per cent chromium, and ¼ per cent vanadium	For aircraft and automobile valve springs and high-grade motor suspension springs
9. *Ball-race steels*— Steel for hardened steel rolls	800–850 Diamond hardness No.	1 per cent carbon and 1–1¼ per cent chromium	For ball bearings and ball races, and hardened steel rolls for making sheet and foil
10. *Magnet steels*— Tungsten magnet	High remanence of the order of 10,500–11,000 c.g.s. units	0·6 per cent carbon and 6 per cent tungsten	For telephone and magneto magnets
Cobalt magnet	High coercive force up to order of 230 c.g.s. units and high permeability	1 per cent carbon, 5–9 per cent chromium, and up to 36 per cent cobalt	For radio loudspeaker, gramophone pick-up, and other magnets in which small overall dimensions are essential
11. *High-speed tool steels*	Maintenance of hard cutting edge even at a dull red heat	0·6–0·7 per cent carbon, 14–22 per cent tungsten, up to 6 per cent chromium, up to 15 per cent cobalt, up to 3 per cent molybdenum, and up to 1½ per cent vanadium	For high-speed lathe turning tools, drills, milling cutters, cold metals, saw inserts, and other severely used tools

Type of Steel	Essential Properties	Essential Analysis	General Uses
12. *Miscellaneous alloy tool steels*	Optimum combination of hardness and extreme toughness to suit different types of duty	0·5–1·5 per cent carbon, up to 10 per cent tungsten, up to 12 per cent chromium, with or without additions of molybdenum, vanadium, and cobalt	For special blanking, forming, and trimming punches and dies, barrel boring tools, wire-drawing dies, cartridge-drawing dies, rivet snaps, chisels, saws, etc.
13. *Special steel having low coefficient of expansion*	Coefficient of expansion of the order of 0·0000015 up to 250° C.	Low carbon and 36 per cent nickel	For standard measuring tapes and numerous details in precision instruments
14. *Special non-magnetic steel having high coefficient of expansion*	Non-magnetic; the coefficient of expansion is of the order of 0·000022	0·6 per cent carbon, 4½ per cent manganese, 12 per cent nickel, and 4 per cent chromium	For non-magnetic rings in electrical generators, and cylinder holding-down bolts, etc., in motor car and aero-engines (coefficient of expansion similar to that of light alloys.)

Type of Steel	Essential Properties	Essential Analysis	General Uses
15. *Rust and acid-resisting steels*—			
12 per cent chromium group	Up to 100 tons/sq. in. tensile and good resistance to rusting	Up to 0·4 per cent carbon and 12 per cent chromium	For cutlery, surgical instruments, knives, and turbine blading
2 per cent nickel–20 per cent chromium group	Up to 60 tons/sq. in. tensile and improved corrosion resistance	0·15 per cent carbon, 2 per cent nickel, and up to 20 per cent chromium	For aircraft construction, motor boat propeller shafts, and other high-tensile shafts and spindles
18 per cent chromium–8 per cent nickel group	50 tons/sq. in. tensile, excellent ductility and resistance to acid attack	Low carbon, 18 per cent chromium, and 8 per cent nickel, with or without other alloy elements with special properties	For tanks, pipes, valves, pumps, etc., in chemical plant, dyeing plant, dairy plant, brewery plant, etc., and numerous domestic, architectural, and decorative applications
16. *Heat-resisting steels*—			
Silicon-chromium	Good resistance to scaling but poor impact strength	0·50 per cent carbon, 3½ per cent silicon, and 9 per cent chromium	For motor car and aero-engine valves
30 per cent chromium	Excellent resistance to scaling but brittle in the cold	Up to 1 per cent carbon and and 30 per cent chromium	For motor car and aero-engine valves, general furnace details, fire grates, mechanical stoker parts, annealing and carburizing pots, enamelling trays. Diesel engine burners mandrel heads for rotary piercing tubes, resistance, wires and grids, etc.
Chromium-nickel High nickel-chromium alloys	Excellent resistance to scaling and excellent strength at elevated temperatures	Low carbon, up to 30 per cent each of chromium and nickel, and up to 4 per cent tungsten Low carbon, 60 per cent nickel and 20 per cent chromium	

TENSILE STRENGTH PROPERTIES OF COPPER ALLOYS AT ELEVATED TEMPERATURES

Material	Condition (all Materials wrought unless stated otherwise)	Tensile Strength (tons/in.^{2})				Elongation per cent on 2 in.				Reference
		Ord. Temp.	300° C.	400° C.	500° C.	Ord. Temp.	300° C.	400° C.	500° C.	
Copper	Annealed	14	10	6	4	60	50	30	20	
Temper Hardened Copper Alloys										
98·75 Cu, 0·75 Ni, 0·5 Si	Temper Hardened	24	21	18	15	35	30	36	3	Cook
99·5 Cu, 0·5 Cr	Cast and Heat Treated	23	—	18	—	25	—	20	—	Corson
97 Cu, 2·6 Co, 0·4 Be	Cast and Heat Treated	40	30 (350°)	—	25 (475°)	10	4 (350°)	—	1 (475°)	Harrington
92·5 Cu, 6 Ni, 1·5 Al	Temper Hardened	44	24	20	16	20	11	5	4	Brownsdon, Cook and Miller
Brasses										
90 Cu, 10 Zn	Annealed	17	11	8	6	56	30	10	15	Price
60 Cu, 40 Zn	Annealed	25	12	6	1	52	42	22	—	Clark and White
57 Cu, 40 Zn, 1·7 Fe, 0·4 Al, 0·5 Pb	Cast	27	14	7	—	15	70	100	—	Bassett
Bronzes										
88 Cu, 10 Sn, 2 Zn	Cast	16	10	7	—	20	8	2	—	
82 Cu, 11 Sn, 7 Ni	Cast	20	16	—	—	20	16	—	—	Mond Nickel Co.
90 Cu, 10 Sn	Annealed	29	16	—	—	63	10	—	—	Bassett
"Chromium Bronze"	Annealed	25	23	18	11	38	40	38	40	J. Stone & Co.
Aluminium Bronzes										
90 Cu, 10 Al	—	38	33	24	12	29	32	41	66	Rosenhain and Lantsberry
80 Cu, 10 Al, 5 Ni, 5 Fe	—	53	—	24 (425°)	—	16	—	37 (425°)	—	Hudson
Other Alloys										
96 Cu, 3 Si, 1 Mn	Annealed	33	24	16	9	43	30	25	20	Bassett
75 Cu, 25 Ni	Annealed	24	18	16	13	37	33	30	20	
68 Ni, 32 Cu	Annealed	37	34	30	23	45	42	35	22	Clark and White
50 Ni, 10 Sn, 40 Cu	Cast	38	—	32	26	1	—	1	1	Mond Nickel Co.
95 Cu, 5 Mn	—	22	21	16	—	37	37	24	—	Heusler

In certain cases the quoted figures have been deduced from graphs of a series of tests given by the various authers.

MECHANICAL PROPERTIES OF CAST ALUMINIUM ALLOYS*

Hiduminium index		23		40 and 42			R.R. 50		R.R. 53		R.B. 53C		"Y"	
		Sand	Die	Annealed	Grade A	Grade B	Sand	Die	Sand	Die	Sand	Die	Sand	Die
Percentage alloy		Magnesium, 2·6–3·3 Manganese, 1·0–1·5 Titanium up to 0·2		Silicon {2·0–2·5–40 / 4·5–5·5–48} Magnesium, 0·6–0·8 Manganese, 0·6–0·8 Titanium, 0·1–0·2			Copper, 0·8–2·0 Nickel, 0·8–1·75 Magnesium, 0·05–0·3 Iron, 0·8–1·4 Titanium, 0·05–0·25 Silicon, 1·5–2·8		Copper, 1·5–2·5 Nickel, 0·5–2·0 Magnesium, 1·4–1·8 Iron, 1·1–1·5 Titanium, 0·02–0·12 Silicon up to 2·0		Copper, 0·8–2·0 Nickel, 0·5–1·5 Magnesium, 0·3–0·8 Iron, 0·8–1·4 Silicon, 2–3 Titanium up to 0·3		Copper, 3·5–4·5 Nickel, 1·8–2·3 Magnesium, 1·2–1·7 Iron up to 0·6 Silicon up to 0·6 Silicon plus iron up to 1·0 Lead up to 0·05 Tin plus zinc up to 0·1 Titanium up to 0·02	
Special properties		High resistance to corrosion		High resistance to corrosion from sea water and marine atmospheres			Good general alloy suitable for large castings		Good qualities at high temperatures combined with excellent bearing properties		High strength alloy suitable for structural components		Good qualities at high temperatures	
Typical applications		Chemical, marine, and architectrual plants		Marine and architectural Castings			Cylinder blocks, cylinder heads, crankcases		Engine pistons, air-cooled cylinder heads		Levers, brackets etc.		Pistons and cylinder heads	
Heat treatment	Solution quench	None required		515–525°C. for 5 hr. Boiling water or oil			—		520–535°C. for 2–4 hr. Boiling water, oil or air blast		520–525°C. for 2-3 hr. Boiling water, oil or air blast		500–520°C. for 6 hr. Boiling water	
	Precipitation quench			—			155–170°C. for 10–16 hr. Hot water or cool in air		150–170°C. for 10–20 hr. or 200°C. for 5–10 hr. Water or cool in air		165–175°C. for 16–20 hr. Water or cool in air		—	
	Ageing			A. 5 days at room temp. B. 10–14 hr. at 155–160°C.			—		—		—		5 days at room temperature	
Tensile test Tons/sq. in.	Ultimate stress	9–11	10–12	8–11	11–16	14–19	11–13	13–16	18–20	21–23	19–22	22–24	14–17	18–20
	0·1% proof stress	4–6	5–7	6–9	7–10	13–18	7·5–10	11–13	—	19–22	18–20	19–21	13–14	14–16
	% elong. on 2"	3–5	5–7	1·5–4	2·5–5	1–2	2–4	4–6	0·5–1·0	0·5–1·5	1–2	1–2	—	2–3

Compression test Tons/sq. in.	0·1% proof stress	—	—	—	—	—	7·5–10	11–13	19–21	23–25	20–23	20–24	—	—
	0·5% przof stress	—	—	—	—	—	11–13	12–14	21–23	25–27	22–25	25–27	—	—
Endurance limit Tons/sq. in.		± 4·8	± 6·8	±4·0–4·5	±4·5–5·0	±4·5–5·0	± 4·5	± 5·8	± 5·5	± 6·9	± 7·2	± 8·4	± 5·8	+ 7·1
Complete cycles × 10^6		10	10	10	10	10	20	20	20	20	20	20	20	20
Brinell hardness number		40–60	50–70	60–70	70–80	85–100	65–75	70–80	124–148	124–148	100–115	110–121	100–125	105–130
Specific gravity		2.65		2.70			2·75		2·75		2·75		2·79	
Specific weight lb./cu. in.		0·095		0·097			0·0985		0·0985		0·0985		0·101	
Young's Modulus (E) 10^6 lb./sq. in.		10·0–10·5		10–10·5			10·0–10·5		10·0–10·5		10–10·5		10–10·5	
0·1% proof stress / Specific gravity		1·51–2·27	1·89–2·65	2·22–3·34	2·6–3·7	4·8–6·65	2·73–3·64	4·0–4·74	—	6·9–8·0	6·55–7·27	6·9–7·65	4·65–5·0	5·0–5·71

* Reproduced by the courtesy of High Duty Alloys, Ltd.

MECHANICAL PROPERTIES OF WROUGHT ALUMINIUM ALLOYS

Hiduminium index		15			33			35			45			DU
		Soft	Half hard	Hard	Soft	Half hard	Hard	Soft	Half hard	Hard	Annealed	Grade A	Grade B	Duralumin
Percentage alloy		Manganese, 1·0–1·5			Magnesium, 2·6–3·3 Manganese, 1·0–1·5 Titanium up to 0·2			Magnesium, 4·5–5·5 Manganese, 0·5–1·5 Titanium up to 0·2			Silicon, 0·5–1·5 Magnesium, 0·5–1·0 Manganese, 0·5–1·0			Copper, 3·5–4·5 Magnesium, 0·4–0·8 Manganese, 0·4–0·7 Silicon up to 0·7 Iron up to 0·7 Titanium up to 0·3
Special properties		Good corrosion resistance low strength			High corrosion resistance medium strength			High corrosion resistance high strength. Difficult workability			Heat treatable, high corrosion resistance to sea water and marine atmospheres			Good strength and ductility for general purposes
Typical applications		Sheets and strips			Sheets, strips, tubes			Chemical, marine, and architectural plants			Extrusions, tubes, sheets, strips, forgings, pressings			Air screw forgings sheets, strips, tubes, bars
Heat treatment	Solution quench	Nil, anneal at 350° C. for cold working			Nil, anneal at 350° C. for cold working			Nil, anneal at 350° C. for cold working			525–540° C. for 2–4 hr. Water			485–505° C. for 2–6 hr. Water or oil
	Precipitation quench										Grade A. 140° C. for 2 hr. ,, B. 155–160° C. for 10–20 hr.			—
	Age										Anneal at 360° C. or 4 hr. Cool in air			5 days at room temp.
Tensile test Tons/sq. in.	Ultimate stress	6·5–7·5	8–11	12–16	15–17	20–22	22–26	20–22	24–28	26–30	7–8	16–18	20–23	25–28
	0·1% proof stress	3·0–4·5	5–10	11·5–14	6–8	16–19	19–23	9–12	19–23	23–27	4–5	10–13	17–19	15–17
	% elong. on 2"	25–40	5–20	3–6	15–25	5–8	3–5	15–25	5–8	2–5	27–20	22–18	14–10	15–20

Compression test Tons/sq. in. — 0·1% proof stress	—	—	—	—	—	—	—	—	—	—	—	—	—
Compression test Tons/sq. in. — 0·5% proof stress	—	—	—	—	—	—	—	—	—	—	—	—	—
Endurance limit Tons/sq. in.	± 3·6	± 4·5	± 5·0	± 8·25			± 8·50			—	± 9·0	± 9·5	± 9·5
Complete cycles × 10^6	10	10	10	10			10			—	10	10	20
Brinell hardness number	27–32	35–48	50–60	40–60	65–85	90–110	45–65	70–90	100–120	30–38	65–80	90–120	100–120
Specific gravity	2·75			2·65			2·60			2·70			2·79
Specific weight lb./cu. in.	0·0985			0·095			0·0935			0·097			0·101
Young's Modulus (*E*) 10^6 lb./sq. in.	10–10·5			10–10·5			10–10·5			10–10·5			10–10·5
0·1% proof stress / Specific gravity	1·09–1·64	1·82–3·64	4·18–5·1	2·26–3·02	6·04–7·15	7·15–8·70	3·46–4·61	7·3–8·85	8·83–10·04	1·48–1·85	3·7–4·81	6·3–7·05	5·37–6·07

MECHANICAL PROPERTIES OF WROUGHT ALUMINIUM ALLOYS

Hiduminium index		R.R. 56				R.R. 59	72	R.R. 77			"Y"
		Solution treated and aged	Solution treated	Annealed	Normalized Annealed	Of test bars forged to 1⅛ in. diam.	B sections sheets	Annealed	Solution treated. Naturally aged	Solution treated. Artificially aged	
Percentage alloy		Copper, 1·8–2·5 Nickel, 0·6–1·4 Magnesium, 0·65–1·2 Iron, 0·6–1·2 Titanium, 0·05–0·15 Silicon, 0·55–1·25				Copper, 1·5–3·0 Nickel, 0·5–1·5 Magnesium, 1·2–1·8 Iron, 1·0–1·5 Titanium up to 0·2 Silicon up to 1·3	Copper, 3·5–4·8 Magnesium, 0·8–1·8 Manganese, 0·3–1·5 Iron up to 0·4 Silicon up to 0·5	Copper, 1·5–3·0 Zinc, 4·0–6·0 Magnesium, 2·0–4·0 Iron up to 0·6 Silicon up to 0·6 Manganese up to 1·0 Titanium up to 0·3			Copper, 3·5–4·5 Nickel, 1·8–2·3 Magnesium, 1·2–1·7 Iron up to 0·6 Silicon up to 0·6 Silicon plus iron up to 1·0 Other impurities up to 0·5
Special properties		High strength and good working properties				High strength at elevated temperatures	High strength natural ageing	High strength properties greater than that of high-tensile steel alloys			High strength at elevated temperatures
Typical applications		Crankcases, connecting-rods, air-screw blades, air-frames, ribs, wing spars, stressed skin covering, cowlings, etc.				Pistons of all classes and general purpose structures	Sections, sheets, strips, tubes	Bars and extrusions			Pistons, cylinder heads
Heat treatment	Solution quench	525–535° C. for 2–6 hr. Hot water				520–530° C. for 2–3 hr. Water, oil, air blast	495–505° C. for 2–6 hr. Water	455–465° C. for 2–3 hr. Water			490–525° C. for 2–6 hr. Boiling water or oil
	Precipitation quench	165–180° C. for 10–12 hr. Water or cool in air				150–170° C. for 15–20 hr. 200° C. for 5–10 hr. Water or cool in air	—	130–140° C. for 15–20 hr. Cool in air			—
	Age	—				—	Natural ageing 3 or 4 days	—			Room temp. for 5 days or boiling water for 2–4 hr.
Tensile test Tons/sq. in.	Ultimate stress	27–30	22–26	12–15	16–19	27–30	28–30	12–14	29–32	33–38	24–27
	0·1% proof stress	21–23	10–13	6–8	7–11	20–22	18–20	4–8	18–21	28–33	14–17
	% elong. on 2"	10–15	15–22	20–25	16–22	10–15	15–20	20–14	21–16	16–10	15–20

Compression test Tons/sq. in.	0·1% proof stress	22–24	12–14	8–10	—	21–23	19–21	4–8	17–21	27–30	—
	0·5% proof stress	27–29	15–17	9–11	—	25–27	24–26	7–9	22–25	32–36	—
Endurance limit Tons/sq. in.		± 10·04	—	—	± 8·73	± 10·3	± 11·5	—	—	± 12·5 to 13	± 9·3
Complete cycles × 10^6		20	—	—	20	20	20	—	—	10	20
Brinell hardness number		121–138	80–100	45–55	70–80	124–148	129–148	45–65	130–140	160–180	100–130
Specific gravity		2·75				2·75	2·76	2·8			2·79
Specific weight lb./cu. in.		0·0985				0·0985	0·0985	0·101			0·101
Young's Modulus (E) 10^6 lb./sq. in.		10·0–10·8				10·0–10·5	10–10·5	10–10·5			10–10·5
0·1% proof stress / Specific gravity *		7·65–8·38	3·65–4·73	2·18–2·82	2·55–4·0	7·3–8·0	6·55–7·27	1·44–2·88	6·45–7·5	10·0–11·8	5·01–6·1

* Ratio for carbon steel, 1·9, 3½% nickel stee 15·71, nickel chrome steel 9·15

CONVERSION TABLES*

CORRELATING DIAMOND AND BALL INDENTER IMPRESSIONS AND NUMBERS WITH ROCKWELL AND SHORE NUMBERS AND WITH TENSILE STRENGTHS

Firth Hardometer Scale Reading 136° Diamond Indenter 30 kg. Load	Diamond Hardness Number	Brinell Impression mm.	Brinell Hardness Number	Approx. Tensile Strength Tons/sq. in.	Rockwell Hardness Number	Shore Scleroscope Number
2·35	1007	2·25	745	—	—	101
2·45	927	2·30	712	—	—	96
2·55	856	2·35	682	—	C65	91
2·65	792	2·40	653	—	62½	88
2·75	736	2·45	627	—	60½	84
2·83	698	2·50	601	132	C59	81
2·90	661	2·55	578	126	57	78
2·98	627	2·60	555	122	55	75
3·05	598	2·65	534	116	53½	72
3·15	370	2·70	514	112	52	70
3·20	543	2·75	495	108	50½	67
3·28	519	2·80	477	104	49	65
3·35	496	2·85	461	101	47½	63
3·43	474	2·90	444	98	46	61
3·50	454	2·95	429	94	45	59
3·58	435	3·00	415	92	C43½	57
3·65	418	3·05	401	88	42	55
3·73	401	3·10	388	84	41	54
3·80	385	3·15	375	82	40	52
3·88	371	3·20	363	80	38	50
3·95	357	3·25	352	76	37	49
4·00	348	3·30	341	74	36	47
4·08	335	3·35	331	72	35	46
4·15	323	3·40	321	70	34	45
4·23	312	3·45	311	68	33	44
4·30	302	3·50	302	66	C32	43
4·35	294	3·55	293	64	31	42
4·40	286	3·60	285	62	30	41
4·50	276	3·65	277	60	29	40
4·55	269	3·70	269	59	28	39
4·60	262	3·75	262	58	27	38
4·65	255	3·80	255	56	26	37
4·75	247	3·85	248	54	24½	36
4·80	241	3·90	241	52	23	35
4·85	236	3·95	235	51	22	34
4·95	230	4·00	229	50	B99	34
5·00	224	4·05	223	49	98	33
5·05	218	4·10	217	47	97	32

* Reproduced by the courtesy of Messrs Thos. Firth & John Brown, Ltd., Sheffield

CORRELATING DIAMOND AND BALL INDENTER IMPRESSIONS AND NUMBERS WITH ROCKWELL AND SHORE NUMBERS AND WITH TENSILE STRENGTHS

Firth Hardometer Scale Reading 136° Diamond Indenter 30 kg. Load	Diamond Hardness Number	Brinell Impression mm.	Brinell Hardness Number	Approx. Tensile Strength Tons/sq. in.	Rockwell Hardness Number	Shore Scleroscope Number
5·10	212	4·15	212	46	B96	31
5·20	207	4·20	207	45	95	31
5·25	202	4·25	201	44	94	30
5·30	197	4·30	197	43	93	29
5·40	192	4·35	192	42	92	29
5·45	187	4·40	187	41	91	28
5·50	182	4·45	183	40	90	28
5·60	178	4·50	179	39	B89	27
5·65	174	4·55	174	38	88	27
5·70	170	4·60	170	37	87	26
5·75	166	4·65	167	36	86	26
5·85	162	4·70	163	35	85	25
5·90	158	4·75	159	34·5	84	25
5·95	155	4·80	156	34	83	24
6·05	152	4·85	152	33·5	82	24
6·10	149	4·90	149	33	81	23
6·20	146	4·95	146	32·5	80	23
6·25	143	5·00	143	32	B78½	—
6·30	140	5·05	140	31·5	77	—
6·40	137	5·10	137	31	76	—
6·45	134	5·15	134	30·5	75	—
6·50	132	5·20	131	30	74	—
—	—	5·25	128	29·5	72	—
—	—	5·30	126	29	71	—
—	—	5·35	123	28·5	70	—
—	—	5·40	121	28	69	—
—	—	5·45	118	27·5	67½	—
—	—	5·50	116	27	B66	—
—	—	5·55	114	26	65	—
—	—	5·60	111	25	63½	—
—	—	5·65	109	24·6	62	—
—	—	5·70	107	24·3	60	—
—	—	5·75	105	24	58½	—
—	—	5·80	103	23·5	57	—
—	—	5·85	101	23	56	—
—	—	5·90	99	22·5	54½	—
—	—	5·95	97	22	53	—

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4 4	9 8	13 12	17 16	21 20	26 24	30 28	34 32	38 37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4 4	8 7	12 11	15 15	19 19	23 22	27 26	31 30	35 33
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3 3	7 7	11 10	14 14	18 17	21 20	25 24	28 27	32 31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3 3	7 7	10 10	13 12	16 16	20 19	23 22	26 25	30 29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3 3	6 6	9 9	12 12	15 15	18 17	21 20	24 23	28 26
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 3	6 5	9 8	11 11	14 14	17 16	20 19	23 22	26 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 3	5 5	8 8	11 10	14 13	16 15	19 18	22 21	24 23
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	3 2	5 5	8 7	10 10	13 12	15 15	18 17	20 19	23 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 2	5 5	7 7	9 9	12 11	14 14	16 16	19 18	21 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 2	4 4	7 6	9 8	11 11	13 13	16 15	18 17	20 19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	11
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Angle.		Chord.	Sine.	Tangent.	Co-tan-gent.	Cosine.			
Deg-rees.	Ra-dians.								
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	·0872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66
25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45
			Cosine.	Co-tan-gent.	Tangent.	Sine.	Chord.	Ra-dians.	Deg-rees.
								Angle.	

INDEX